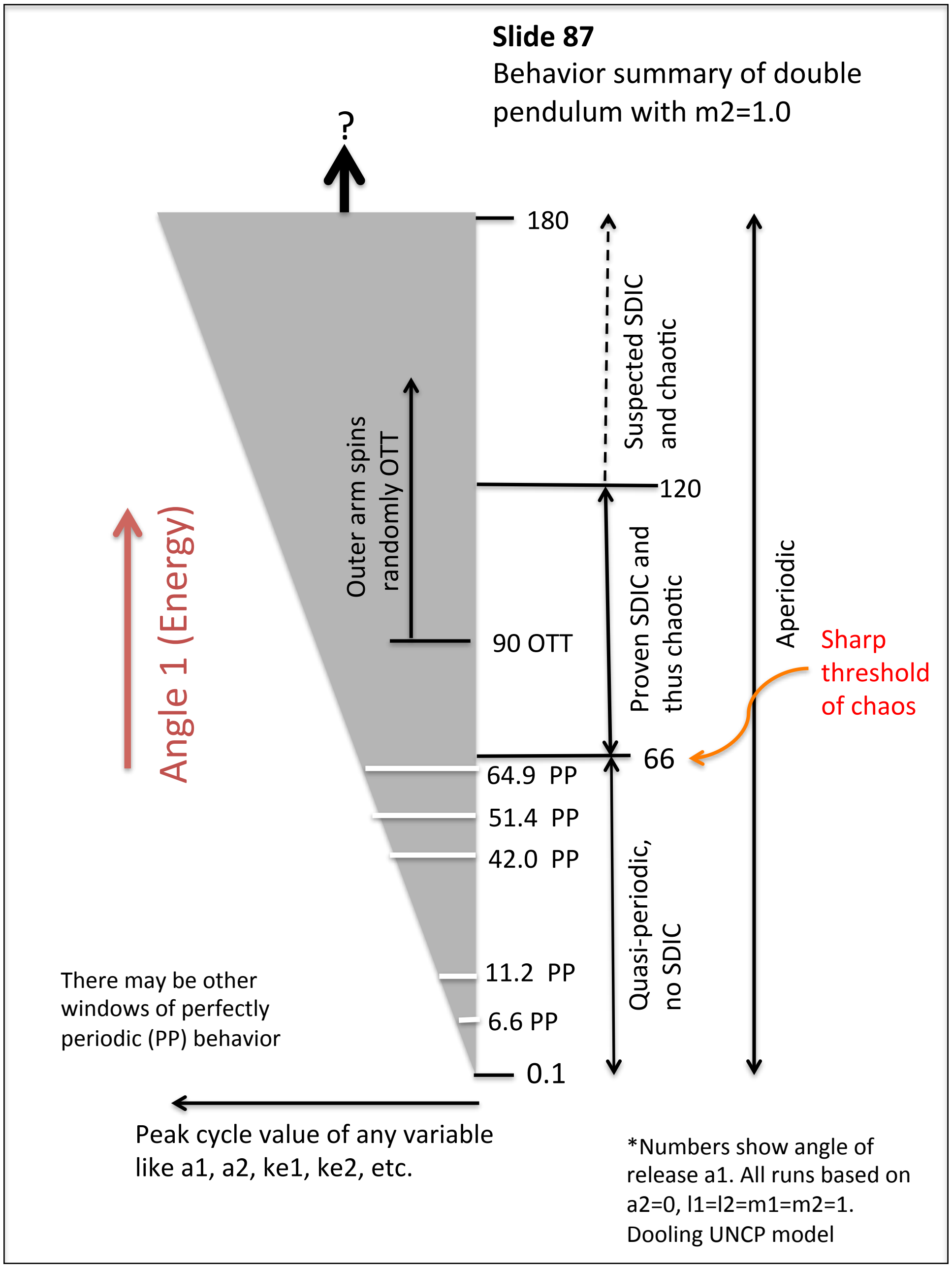
Chapter 9 Double Pendulum Analysis (Part 4)

Outline

9.1 Overview of behavior

Slide 87 summarizes the behavior of the double pendulum with m2=1.0. Generally it's the same pattern of behavior as that seen with m2=0.5, but of course all the numbers are different. The most notable difference is that with m2=0.5 the system was perfectly periodic, chaotic, AND able to go over the top all at about the same energy level (a1 about75 degrees). Here the system was perfectly periodic and chaotic at nearly the same energy level (a1 about 65 degrees), but it wasn't able to go over the top until a higher energy was reached.

This section also briefly studied the transition from quasi-periodic to chaotic operation to see if any dramatic or notable differences in the system behavior (waveforms) occurred.

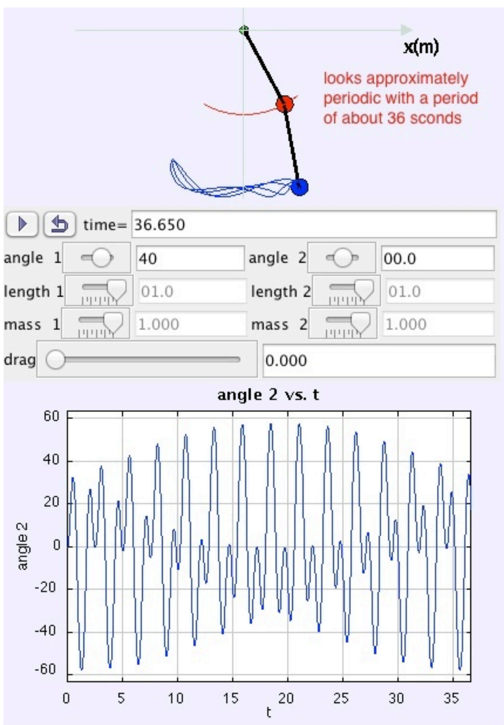


9.2 Quasi-periodic behavior

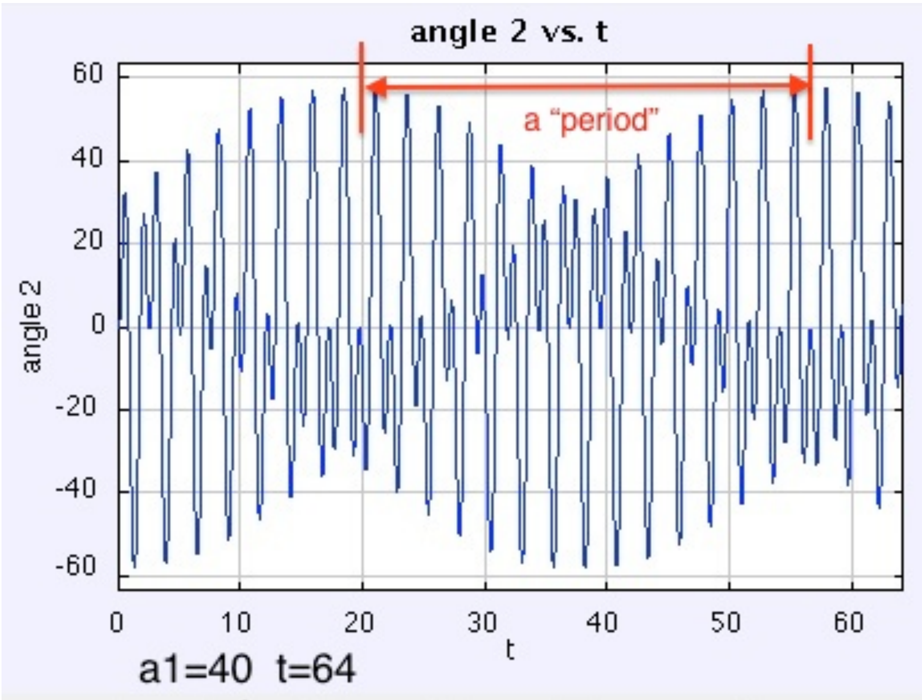
I have judged that the double pendulum is quasi-periodic when it has little energy because the bob was released at a low angle. This conflicts with other authors who say the double pendulum is periodic at low energy. This section provides support for my claim.

I’ve not seen the term “quasi-periodic” defined in the technical literature but it apparently means behavior that is somewhat periodic but not perfectly periodic. Its like a dance whose general movements repeat time after time but the twists and turns aren’t all the same shape and the steps aren’t always the same size. A ballet is a good example of quasi-periodic behavior because the moves during any given performance (i.e.: period) will not be exactly the same from one performance to another. Note that the moves within a performance can be quite complex as can the movements of the double pendulum during one if its “periods”. I use quotes because a period is arguably not a true period unless the moves repeat exactly, not approximately. If the moves repeat exactly I use the term “perfectly periodic”. Others may use the term period casually, thus creating confusion.

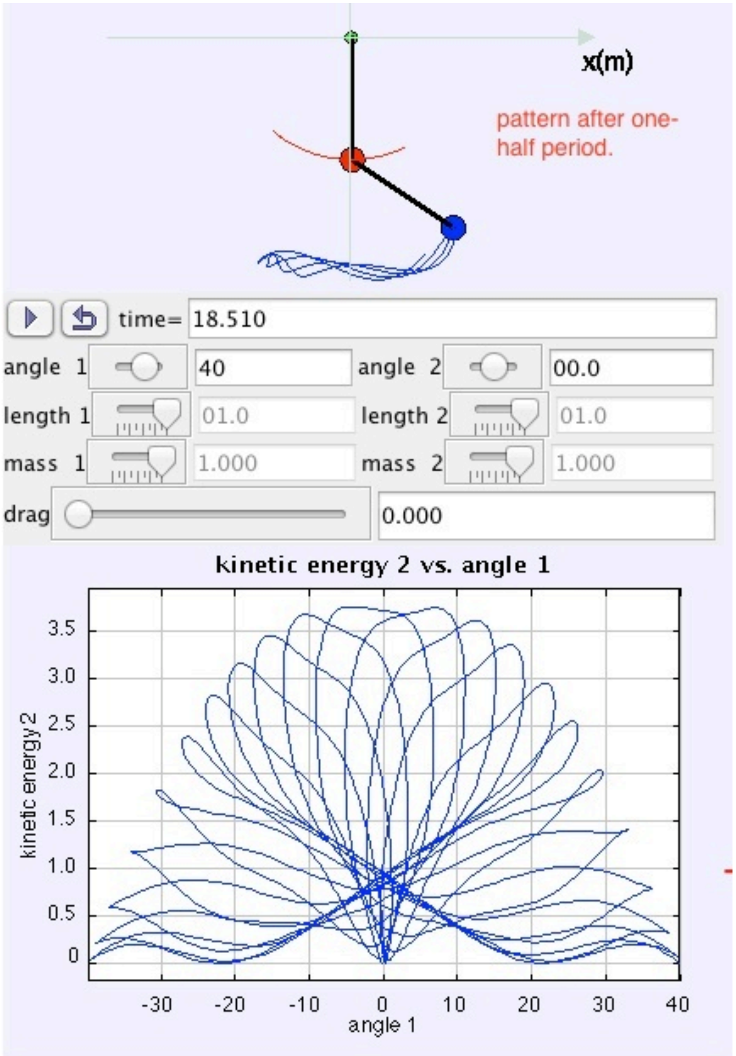
The screenshot below shows a low energy run where the red bob was released at 40 degrees. This run used the Dooling simulation model and aside from a1 all parameters were left at their default values. This waveform appears to complete its pattern or one period after about 36 seconds. Again all these results assume that the Dooling simulation model I used is accurate and doesn’t have some artificial characteristics that makes the patterns drift.



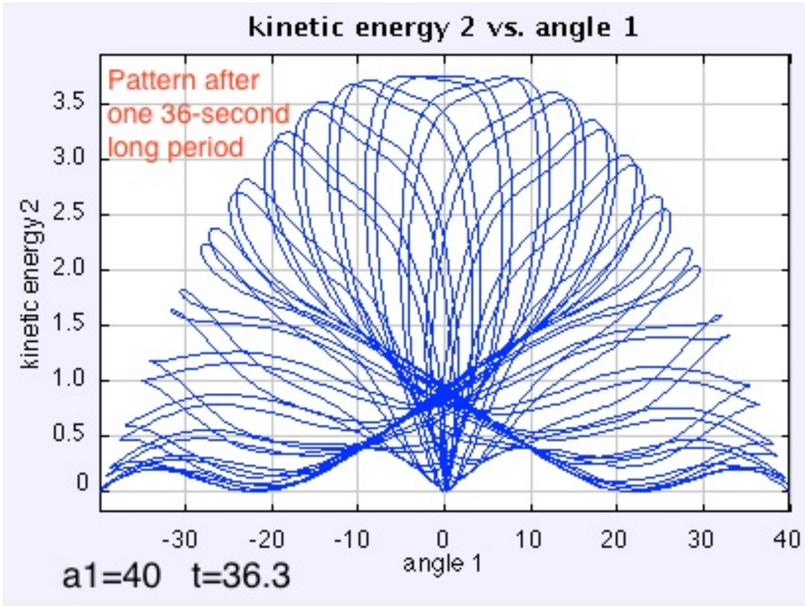
The screenshot below shows the situation after 64 seconds and seems to confirm that this is somewhat periodic operation with a period of about 36 seconds. However its not obvious whether the waveforms, the heights of the peaks is exactly identical from one period to the next. A different type of plot is needed to show that, namely a partial phase space plot that uses only two of the four variables needed for a full phase space plot. Any two will do. I generally plot a1 versus ke2. The phase space plot allows the trace to be compared over a long period of time to better judge if the pattern repeats.



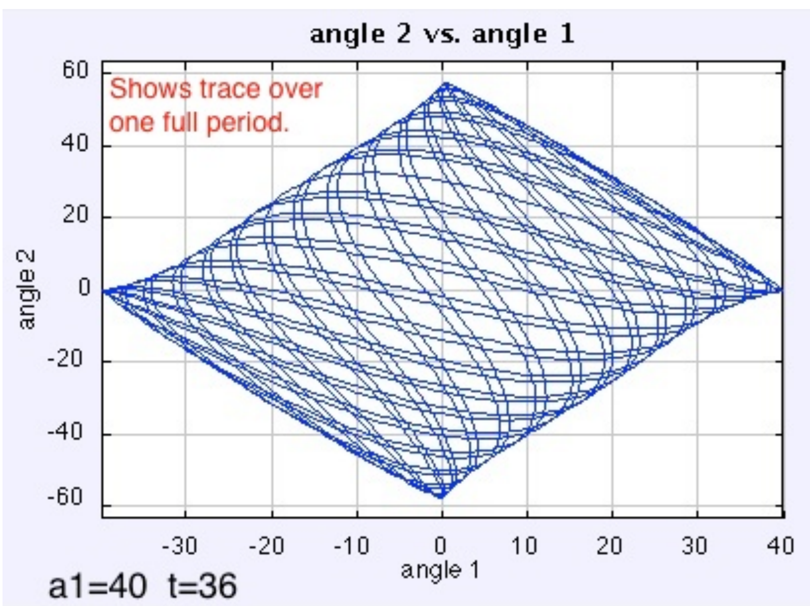
The screenshot below plots angle 1 versus ke2 (a proxy for the speed of the outer bob) over 18 seconds or one half period.



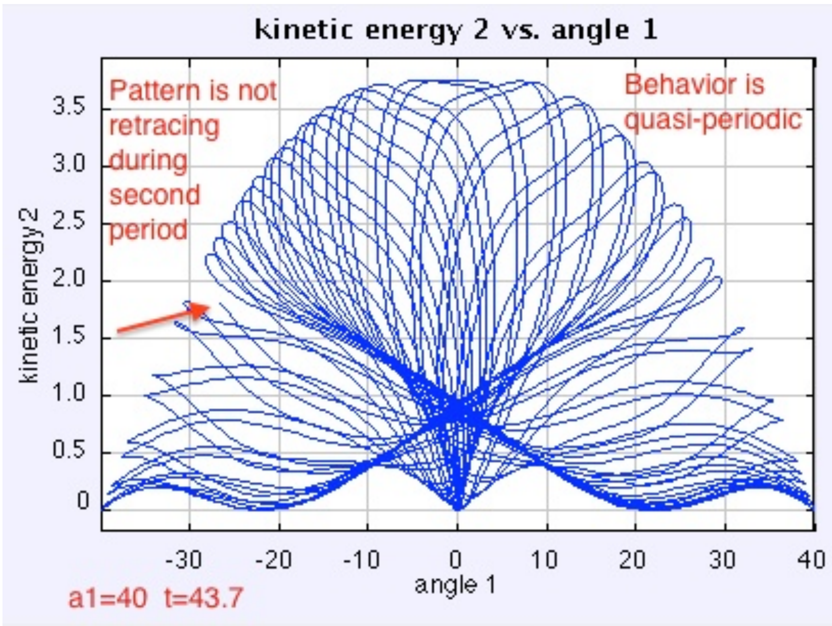
The screenshot below shows the trace over a full 36-second period.



The screenshot below also covers one full period but shows what it looks like if a1 is plotted against a2.

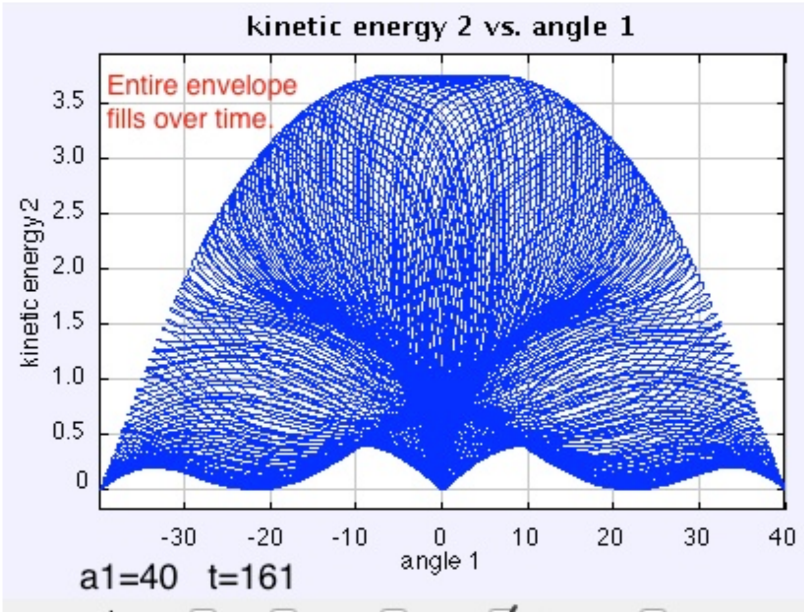


If the system is <u>perfectly</u> periodic this pattern should repeat <u>exactly</u> every 36 seconds thereafter. The new trace would lie exactly atop the existing ones. However the following screenshot shows that as the second period starts the new trace is not exactly retracing the above pattern. Instead it is drawing a somewhat offset pattern as indicated by the arrow. I say the values are drifting. If this system exhibited period doubling –which I haven't been able to detect- then this third line might indicate the beginning of a period-2 oscillation. If so we should eventually see a pattern with four lines that repeats every 72 seconds.



However that does not happen. Instead each new pattern is slightly offset from the prior ones, and will eventually fill in the entire envelope as shown below. Its possible this 161-second long run is showing something like period-4 or period-8 oscillation but I am assuming that's not the case. Instead I assume this is quasi-periodic operation where the exact same waveform with the exact same heights on every peak will never repeat. As such there are no true periods so this is aperiodic behavior and more specifically a subcategory of aperiodic operation called quasi-periodic. The other category is called chaotic.

The size of this envelope increases as the energy within the system is increased. No variable can ever go outside this envelope. In this case the red bob can never go higher than the angle (a1=40 degrees) it was released at since it lacks the energy to do so. However for the blue bob to move and have kinetic energy it must take energy from the red bob so when blues kinetic energy peaks red must loose all its potential energy. We see that at the top of the envelope.



Every other low energy run I've made produces a similar diagram in the sense that there is some general repetitive pattern of movement but the patterns "drift" in value over time. The waveform peaks –like the value of a1 or ke2- eventually touch on all possible values, which is what also happens in chaotic behavior and makes the iconic bifurcation for the logistics equation gray after a sequence of period doubling. The fact that the double pendulum produced these drifting, quasi-periodic plots even at energies as low at a1=.1 degree strongly suggests there is no period-1 oscillation at low energy in the double pendulum whereas there is period-1 oscillation at low driving force (the assumed equivalent of energy) in the bifurcation diagram for the logistics equation.

Quasi-periodic behavior in the double pendulum spans a wide range. At low energy its very close to being perfectly periodic. One can clearly see the dance repeat. At high energy –just short of what could be officially called chaotic- there is almost no discernable pattern, but there is some. Things can be more or less quasi.

Clearly a certain degree of prediction is possible during quasi-periodic operation. If you have watched a pattern approximately repeat a few times you can assume it will continue to do so. The moves will be similar but the size of the steps will differ a bit.

Could quasi-periodic actually be perfectly periodic with a long period? That will be addressed later.

9.3 Perfectly periodic operation

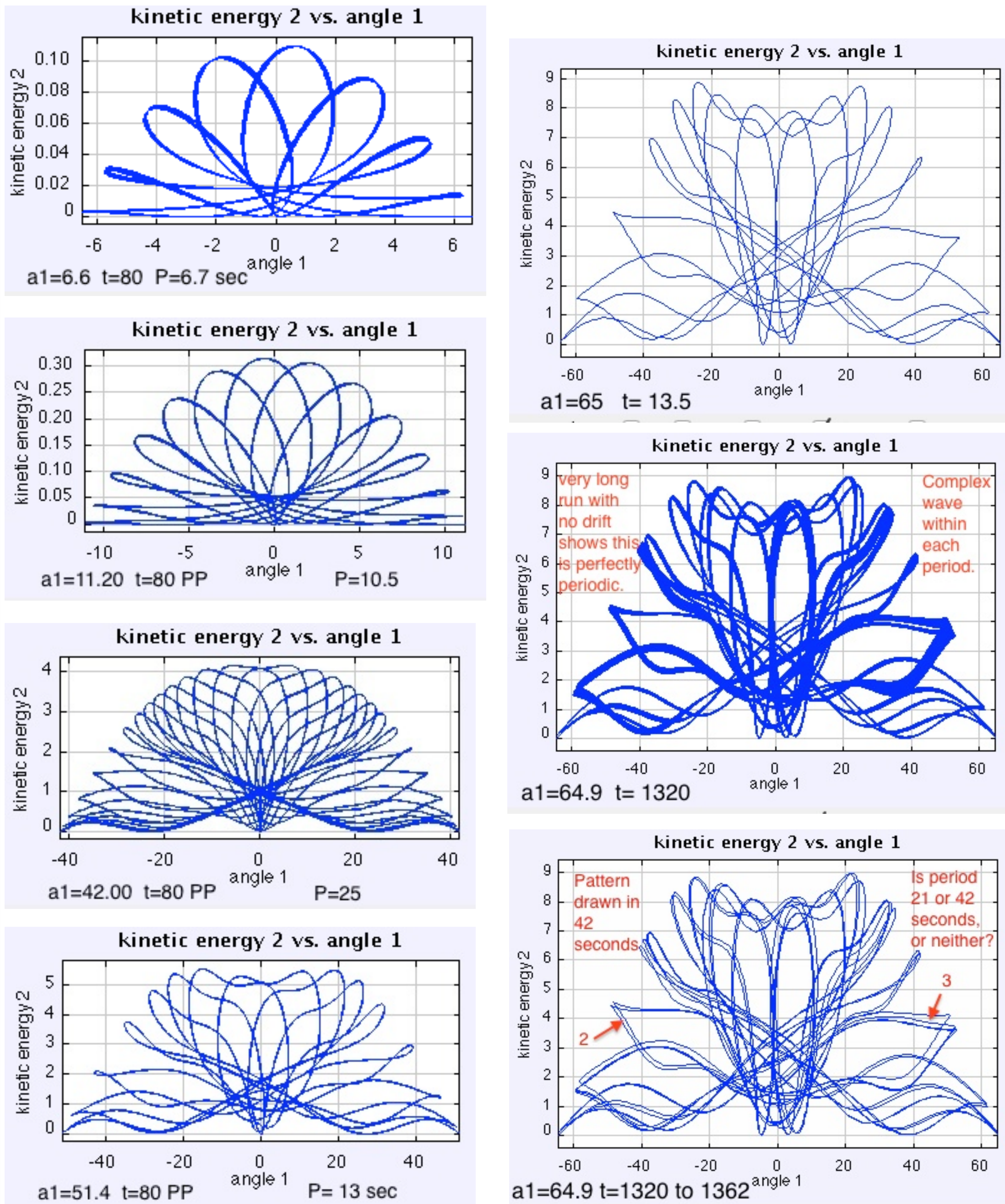
This section describes instances of perfectly periodic operation with m2=1.0

Judgments about whether a given run is perfectly periodic or not are somewhat subjective on the authors part. Generally if the trace stays with a very narrow band over a reasonably long time I call it perfectly periodic. However I've found that some runs which seemed perfectly periodic for say 100 or 200 seconds -during which the pattern retraced almost exactly- eventually turned out to be quasi-periodic if observed over a very long time.

With that in mind Slide 95 shows all the runs that appeared perfectly periodic during the times observed. In most cases m2 was left at its default value of 1.0. It is suspected that a through search would have turned up more than these five.

All with a2=0 m2=1.0
L1=L2=m1=1 drag=0

Slide 95
Perfectly Periodic runs
with m2=1.0



The panels at right provide more detail about the runs at and near angle 65. The pattern at top right was found to redraw fairly accurately several times so runs were made just above and below 65 degrees to fine tune a1 so the pattern would retrace even more accurately. The best result was produced at 64.9 degrees. In a very long run lasting 1320 seconds this pattern was seen to retrace within a very narrow band as shown in the second image down. Perhaps additional fine-tuning would have narrowed the band still more. Nonetheless this run at 64.9 was judged to be perfectly periodic for all practical purposes. The plot was cleared at t=1320 and allowed to continue until t=1362. During that time the pattern was redrawn about twice suggesting the system had a period of either 21 or 42 seconds. I have no explanation for why the patterns differ a bit each time they are drawn while still staying within the band. Its possible the system is indeed perfectly periodic but has a period significantly longer than 21 or 42 seconds. Within that longer period it might draw a series of slightly offset patterns thus creating the band, and then exactly repeat that series over and over. Alternately the trace may wander randomly within that band and never exactly follow the same path twice, as seems to be the case with the Lorenz strange attractor

Its possible that the arms produce perfectly periodic operation when synchronized so their swings follow integer ratios like 1:1, 1:2, 1:3, 2:3 and the like. A deliberate, and successful, attempt was made to get them to swing in 1:1 and 1:2 ratios by experimenting with different initial conditions. Its described just below and did result in a simple perfectly periodic oscillation. This topic merits further research.

It was very difficult to judge the period of these oscillations meaning the time it took to draw a pattern that would repeat. This partly had to do with the fact that the ke2 plots are not the same as regular phase space plots because ke is always a positive number. Thus the negative values which ideally should be below the line are reflected above it. Take the values I picked as periods –and marked on some of these plots with notations like P=6.7 sec- with that in mind.

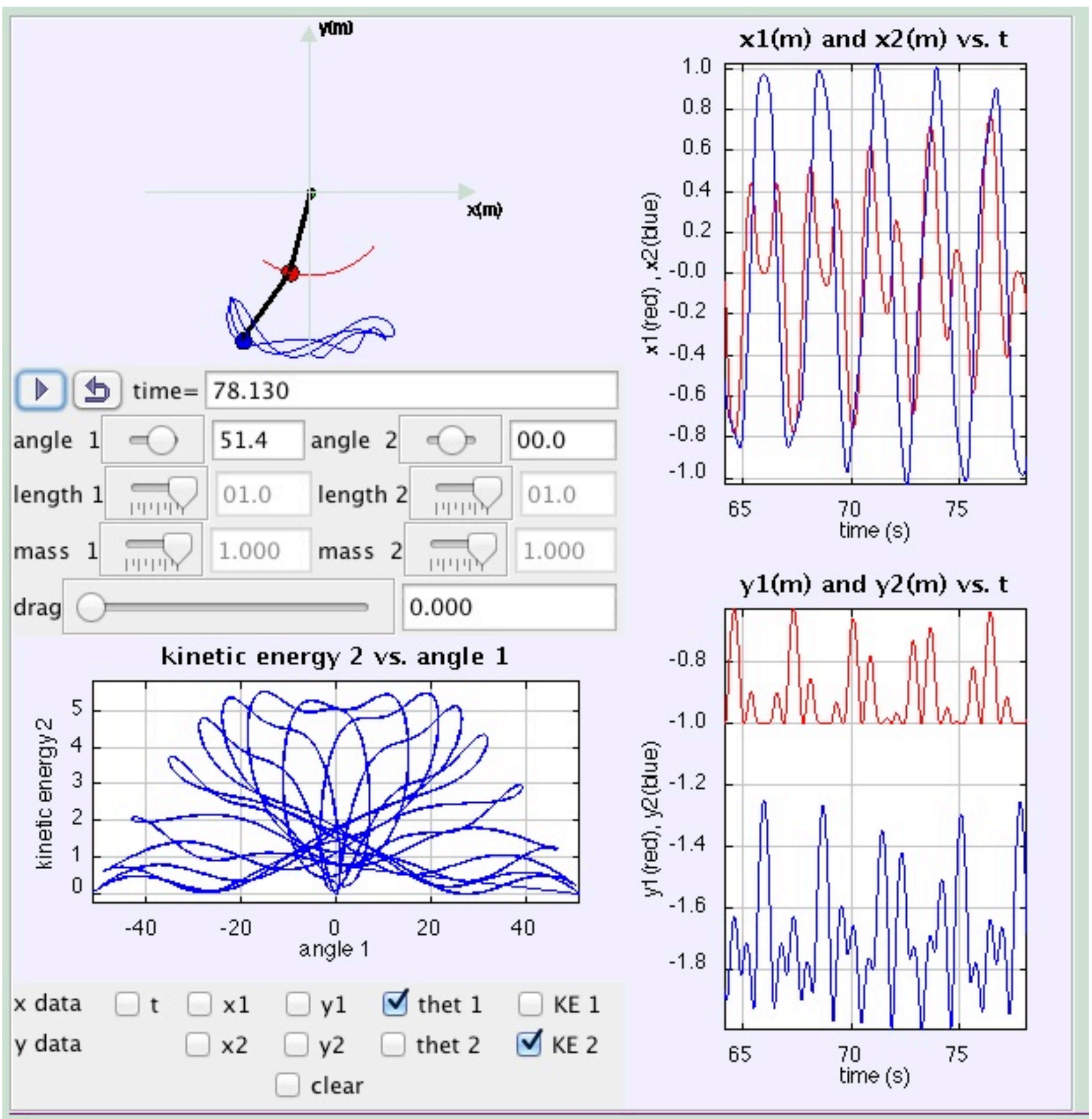
Now we put these perfectly periodic runs in context relative to what happened when the energy level was between the values that gave perfectly periodic behavior. Figure 13 shows some windows of perfectly periodic operation which were discovered with default initial conditions except for the release angle a1, which was raised from 6.6 degrees to 51.4. The runs shown at left were close to perfectly periodic for 80 seconds. The runs on right show that between these energy levels the system was quasi-periodic. The upper right plot shows that the system traced some patterns more often than others. This phenomena was not explored further. This slide provides the main data in support of the general finding that the double pendulum is quasi-periodic over the vast part of its operating range from low to high energy, except for very narrow windows where its perfectly periodic.

This is perhaps analogous to the periodic windows in the bifurcation diagram for the logistics equation and Rosller system, which suggests a common root cause.

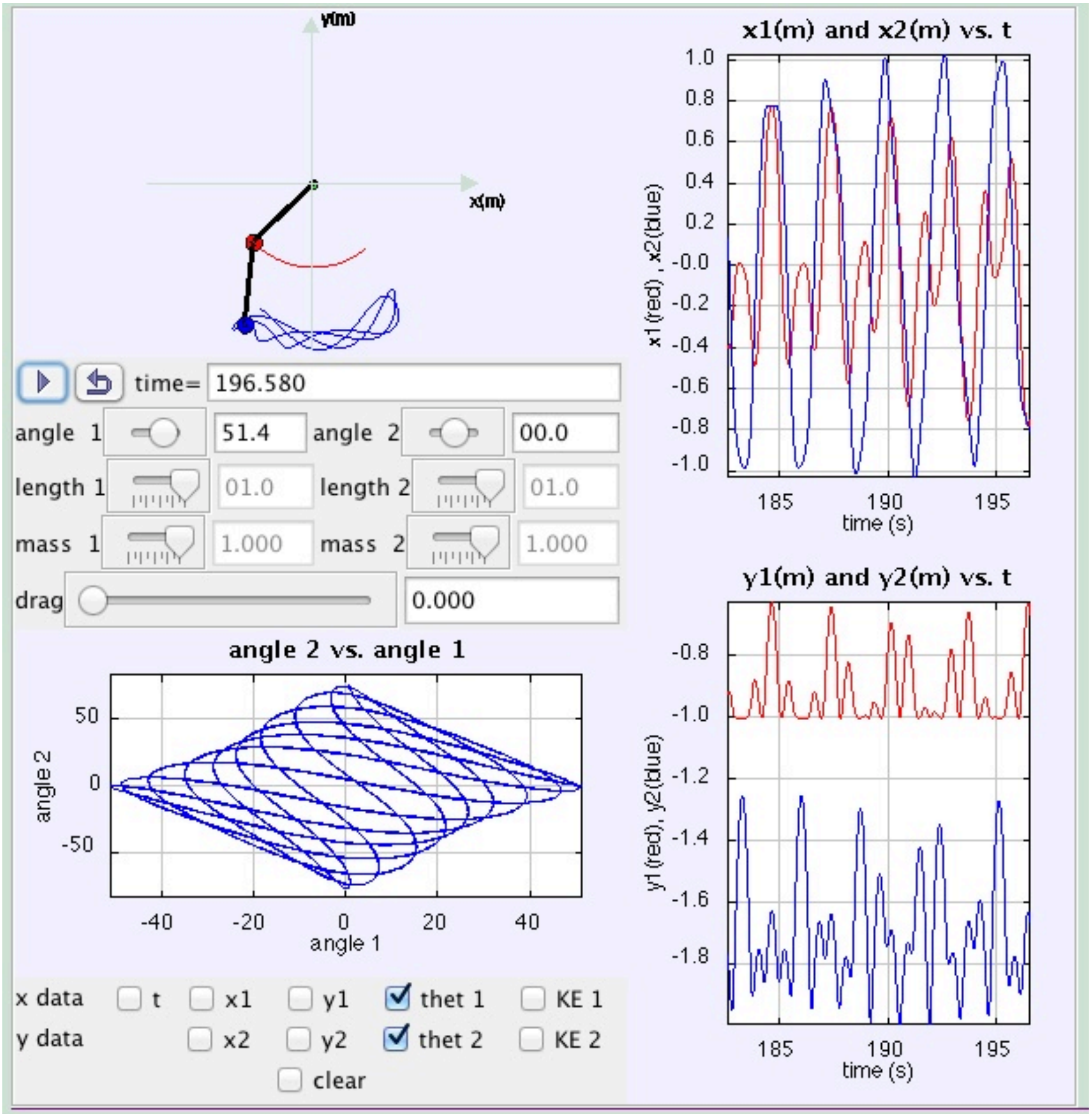
Angle 51.4 details:

By accident one run producing a beautiful almost symmetrical plot indicating what looked like nearly perfect periodic behavior. See the screenshot below (T=78.13). It was discovered when the red pendulum was released from an angle of 51.4 degrees and mass2 was left at the default value of 1.0. Because mass 2 was set at 1.0 not 0.5 the first two runs below differ from most the rest in this section. The trace went round and round this pattern time after time retracing the same line indicating this behavior was periodic not only as to wave shape but also the values accurately repeated. The waveform was obviously complex during that period.

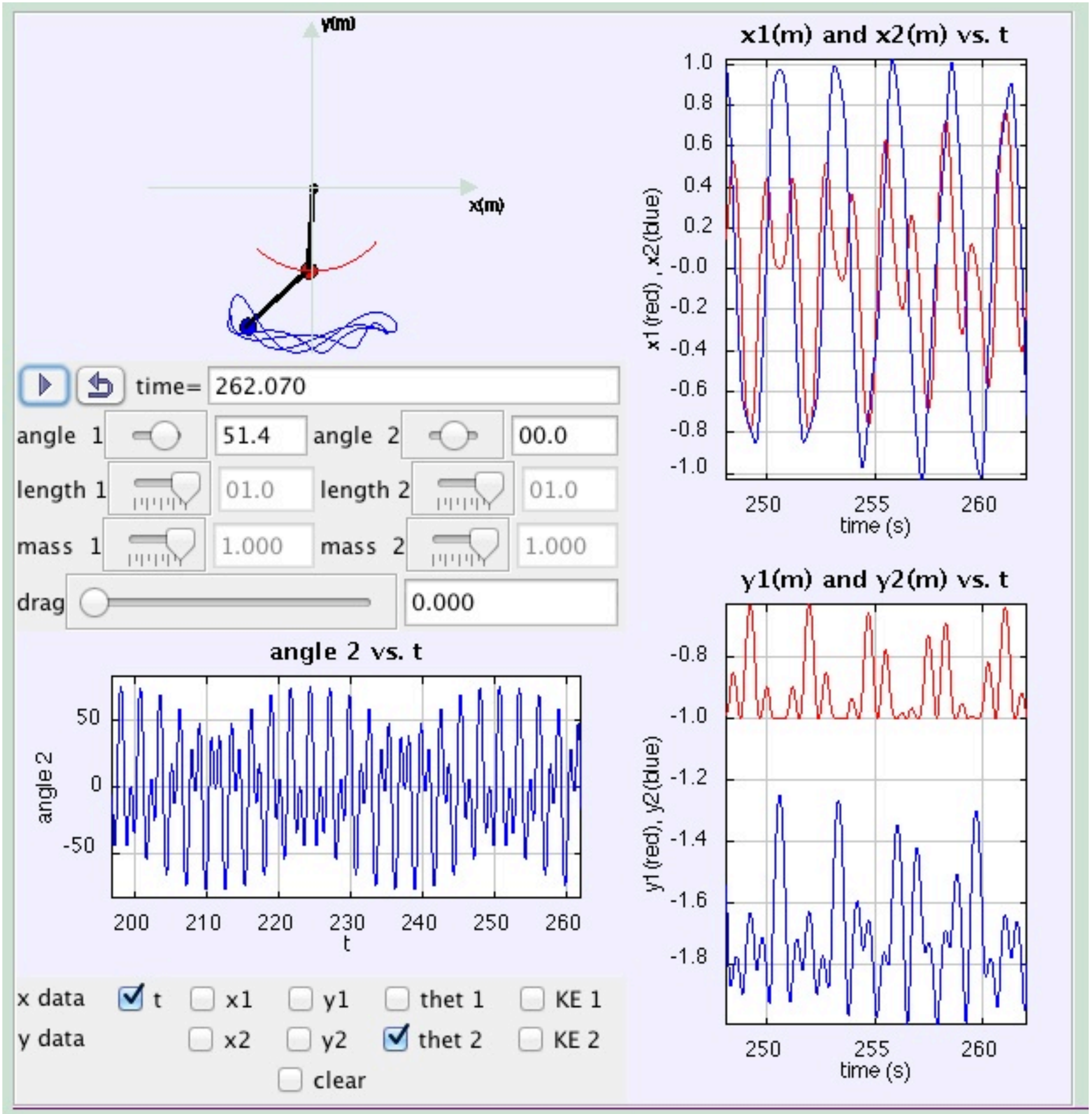
We haven't found many example of near perfect periodic operation in our explorations of the double pendulum so this one is worth examination. Is it fragile in the sense that any small change in the energy level or one of the so called constants like pendulum mass will destroy it?



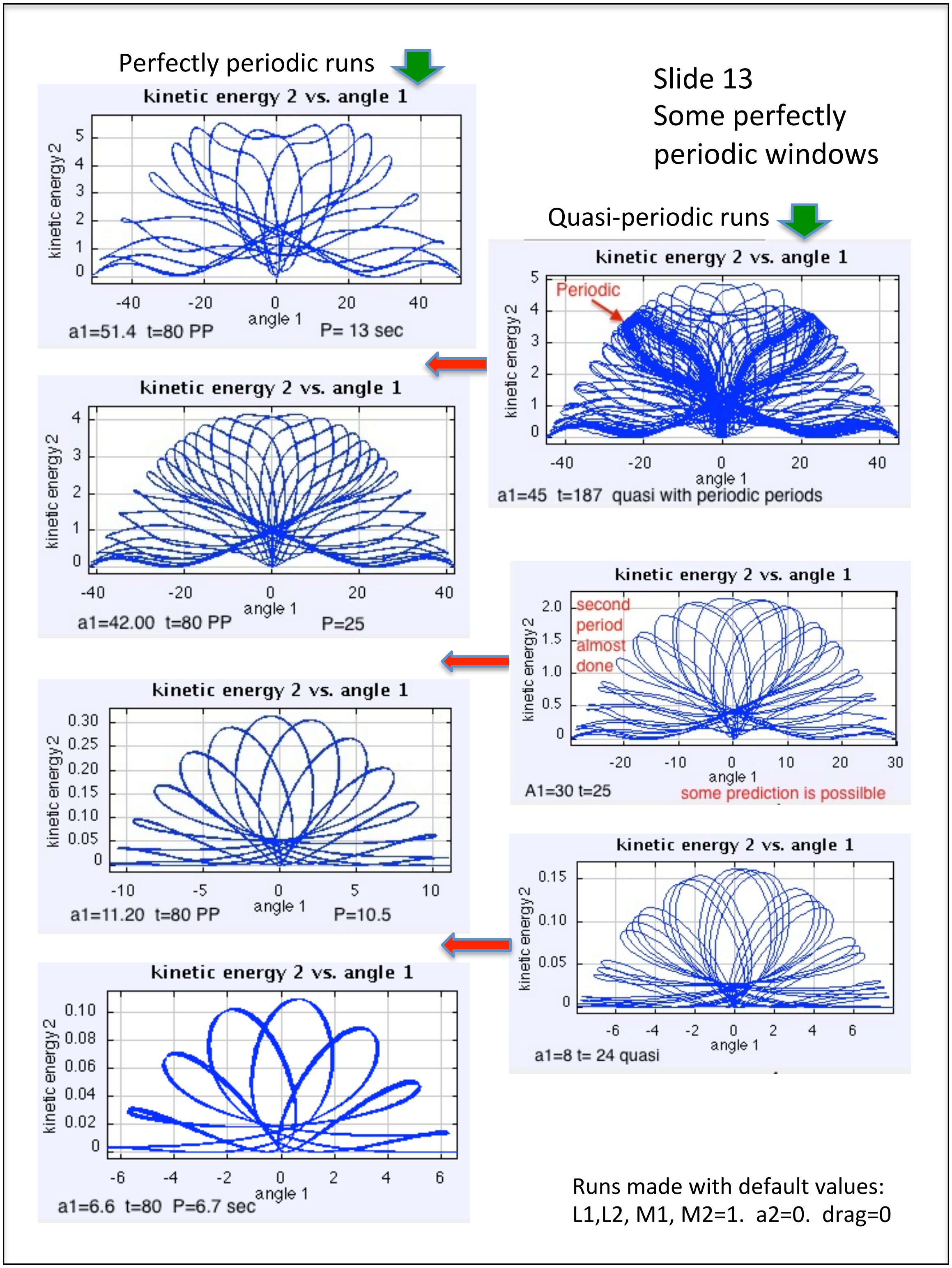
The screenshot below (T=196.58) was the same run but showing the relation between a2 and a1. Again the trace repeated took virtually the same path. Its not perfectly periodic its sufficiently close to call it so. A very small adjustment in angle 1 would probably yield perfection.



The screenshot below (T=262.07) was taken later in this same run. It plots angle 2 over time to expose determine the period of this oscillation. Measuring between the major crests, shows that the period of this behavior is about 25 seconds after which the trace begins retracing the same pattern.

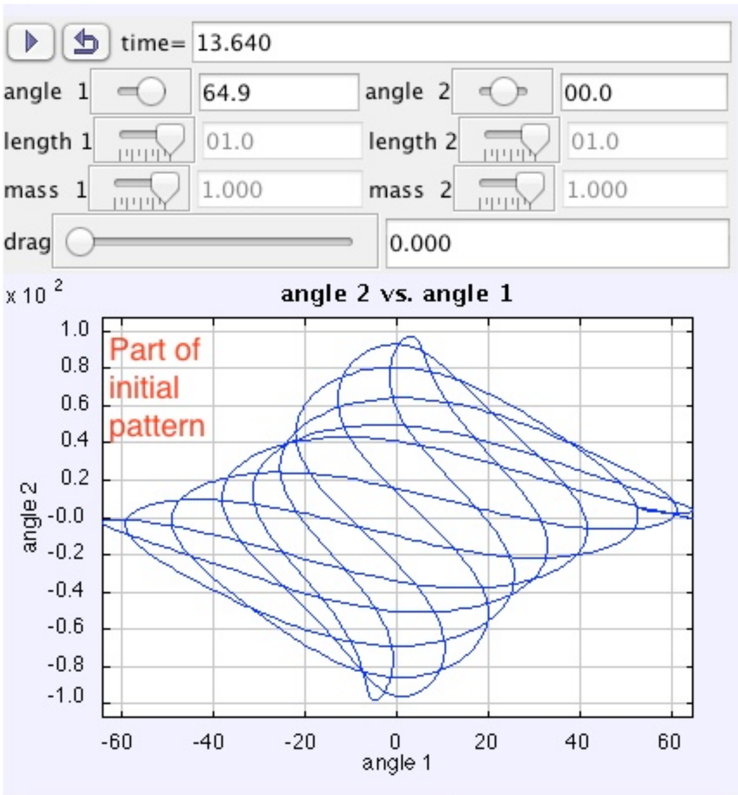


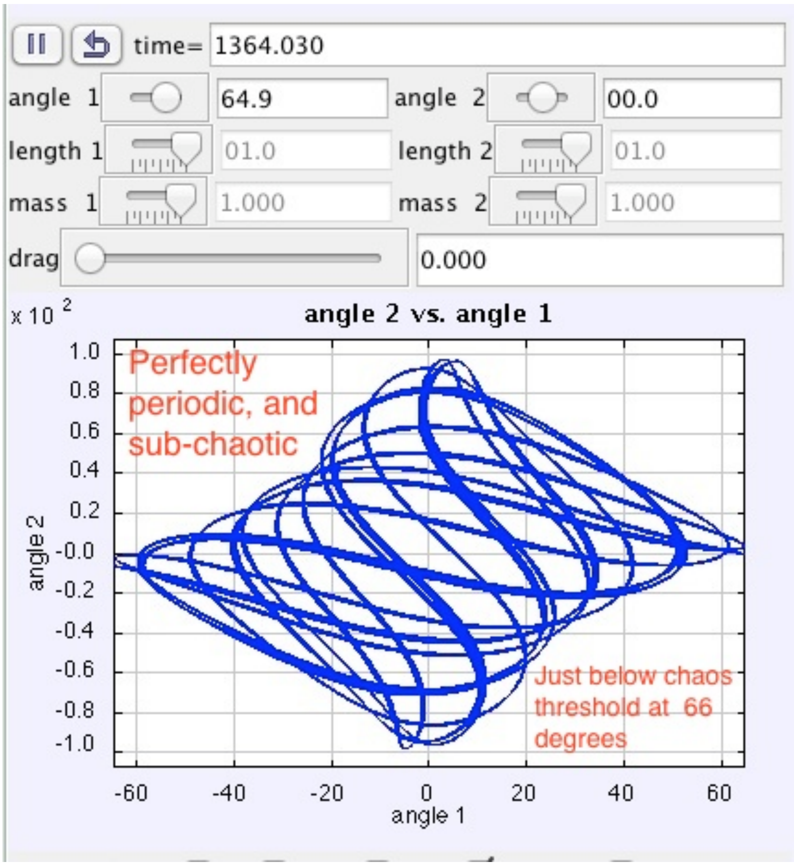
Slide 13 shows that behavior was quasi-periodic between the narrow windows of perfectly periodic operation.



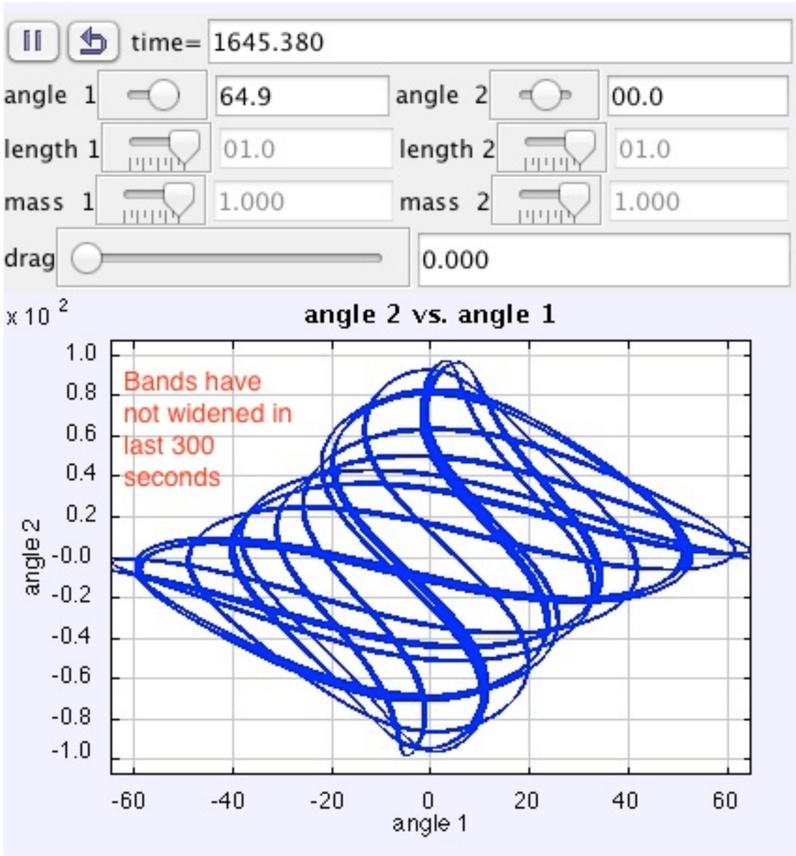
angle 64.9 details: There four screenshots demonstrate that the system was very close to perfectly periodic at 64.9 degrees. The same was true at 65.5 degrees and presumably values in between. It took about 15 seconds to complete the initial

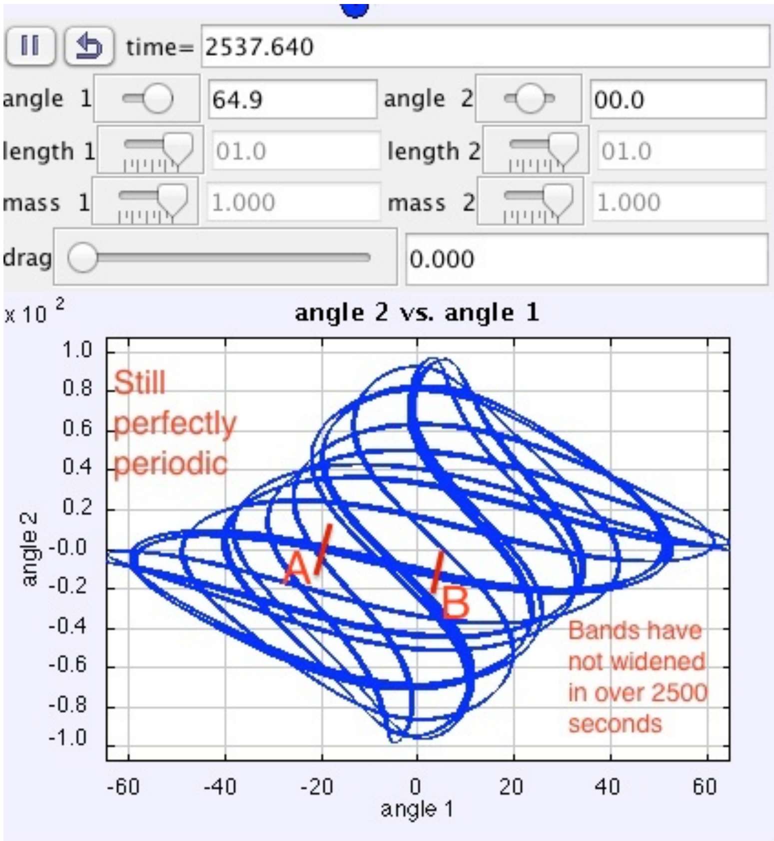
pattern. The last three screenshots show that once the band had widened to a certain amount it didn't widen any further. There is more analysis on this later.





time= 1364.030
angle 1
64.9
angle 2
00.0
length 1
01.0
length 2
01.0
mass 1
1.000
mass 2
1.000
drag
0.000
x 10 2
angle 2 vs. angle 1
Perfectly periodic, and sub-chaotic
Just below chaos threshold at 66 degrees
angle 2
angle 1
1.0
0.8
0.6
0.4
0.2
-0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-60
-40
-20
0
20
40
60


time= 1645.380
angle 1
64.9
angle 2
00.0
length 1
01.0
length 2
01.0
mass 1
1.000
mass 2
1.000
drag
0.000
x 10 2
angle 2 vs. angle 1
Bands have not widened in last 300 seconds
angle 2
angle 1
1.0
0.8
0.6
0.4
0.2
-0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-60
-40
-20
0
20
40
60



9.4 SDIC testing

This sub-section reports the results of testing for SDIC with m2=1 in order to see how they differ from SDIC tests done with m2=0.5 in Section 8.2.

The table below shows when SDIC effects were first noticed. There was no SDIC evident in runs where a1 was below 66 degrees. At and above 66 degrees all runs exhibited SDIC and were therefore chaotic.

Angle a1 (With A2=0, m1=m2=l1=l2=1)	**Time at which waveforms first diverged enough to notice SDIC effects**
20	None by 120 seconds
40	Minor effects seen at 1100
60	None by 250
65	None by 560 (was perfectly periodic)
66	91 (System first became chaotic)
67	71
70	42

80	53
100	23
105	14
110	12
120	7.5

This table presents some additional data that might be useful in discovering the root cause of chaos in the double pendulum, namely the time at which SDIC effects first occurred, or more precisely when I was first able to notice them visually. However its entirely possible, and I think probable, that the waveforms began to diverge right after the simulation run began. If so they grew exponentially at about the time noted in the table and became visually obvious. This seemingly exponential growth, as opposed to a sudden jump, is I think key to understanding the root cause. I want to say the system has reached some sort of tipping point in the phase relationships between all the variables, but I can't picture it.

To highlight this concept of exponential growth I examined the waves more carefully and marked with arrows – in the screenshots below- some very minor differences that occurred before the times I recorded in the table.

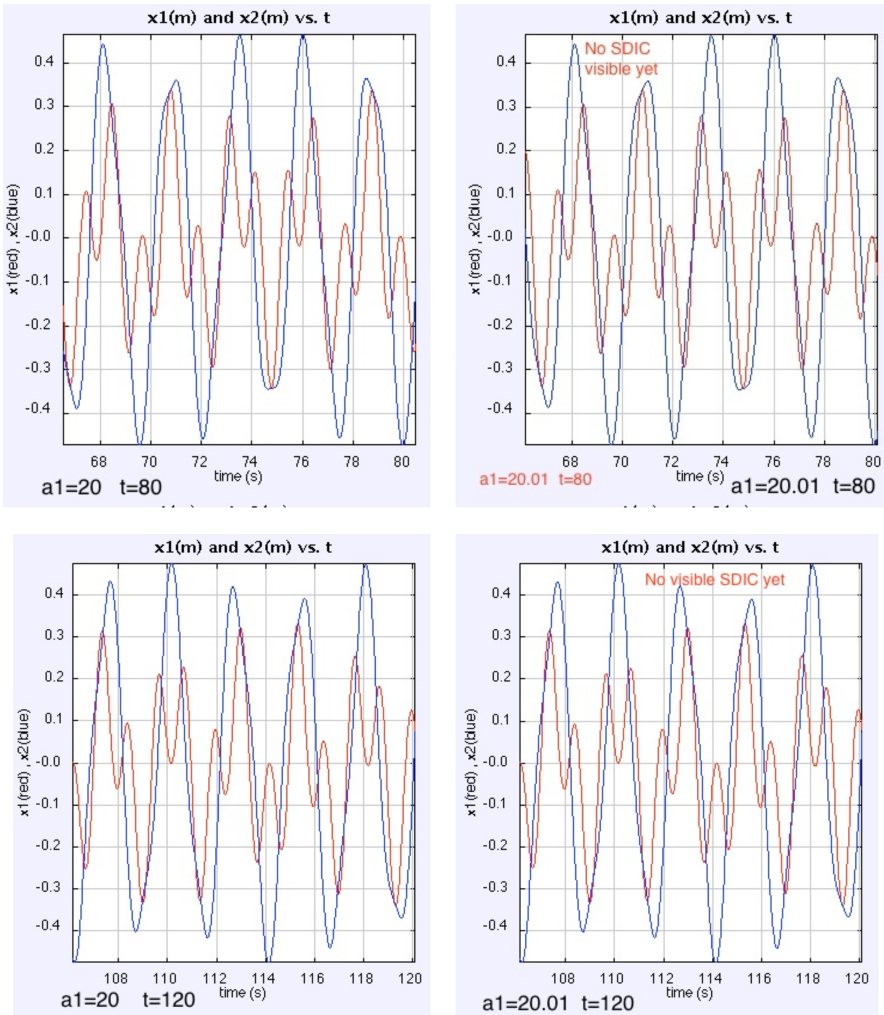
I'll say this again later but I suspect that plotting the waveforms for sheer force in the middle pivot alongside the waveforms of the other variables should be done for a few seconds before and after the times in the table, and then examined carefully to see what is changed and why. Comparisons would be made for the base run and then one with slightly different initial conditions. In addition comparisons would be done between a run that was sub-chaotic, (like the 60 degree run) and one that was chaotic like the 66 degree run.

The series of slides below contain screenshots of the runs used to produce the table above. Usually the left images show the base run and the right ones show the results of having a slightly different initial condition, namely setting a1 one hundredth of a degree higher. Ideally all the runs would have been equally long at low energy to see if SDIC emerged after a long time. That wasn't convenient.

20 degree runs:

Slide 73
Development of SDIC at angle 20

SDIC had not become evident by 80 seconds and was still not evident when run ended at 120 seconds. It might occur much much later.

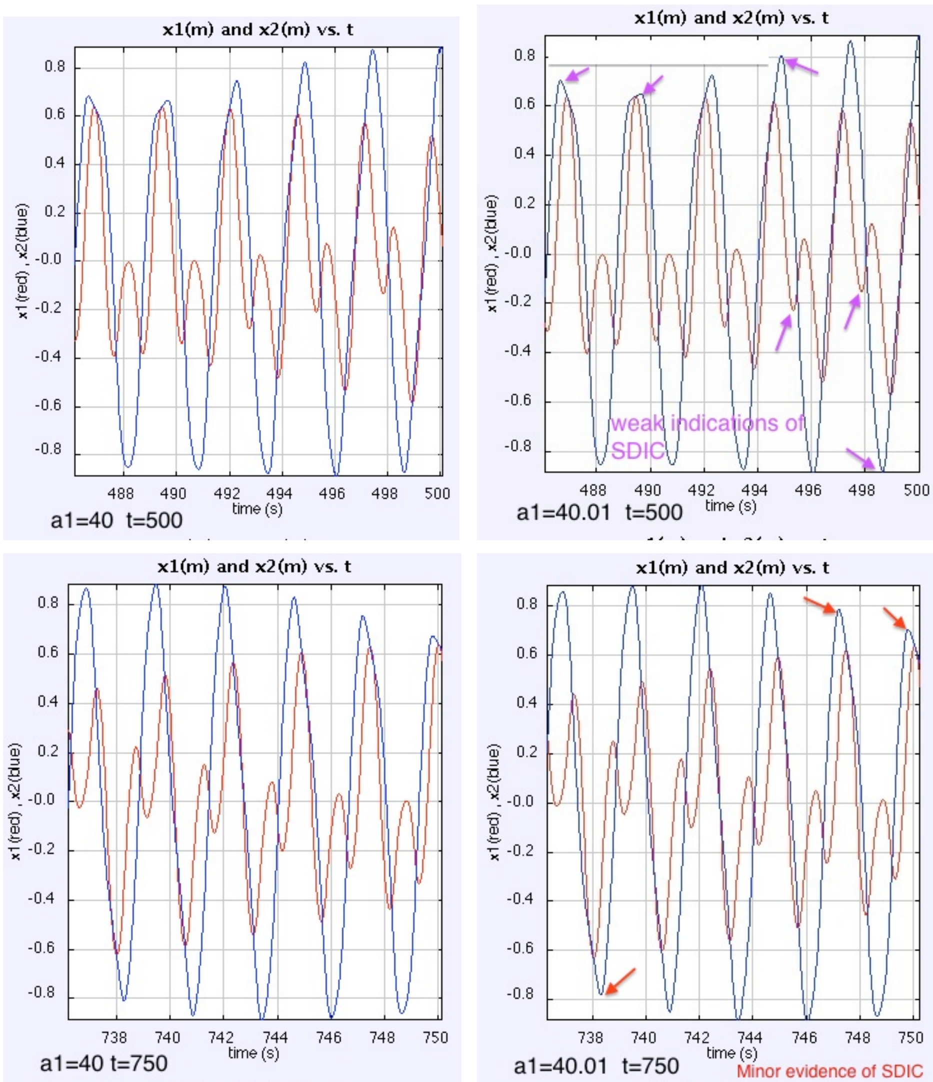


40 degrees:

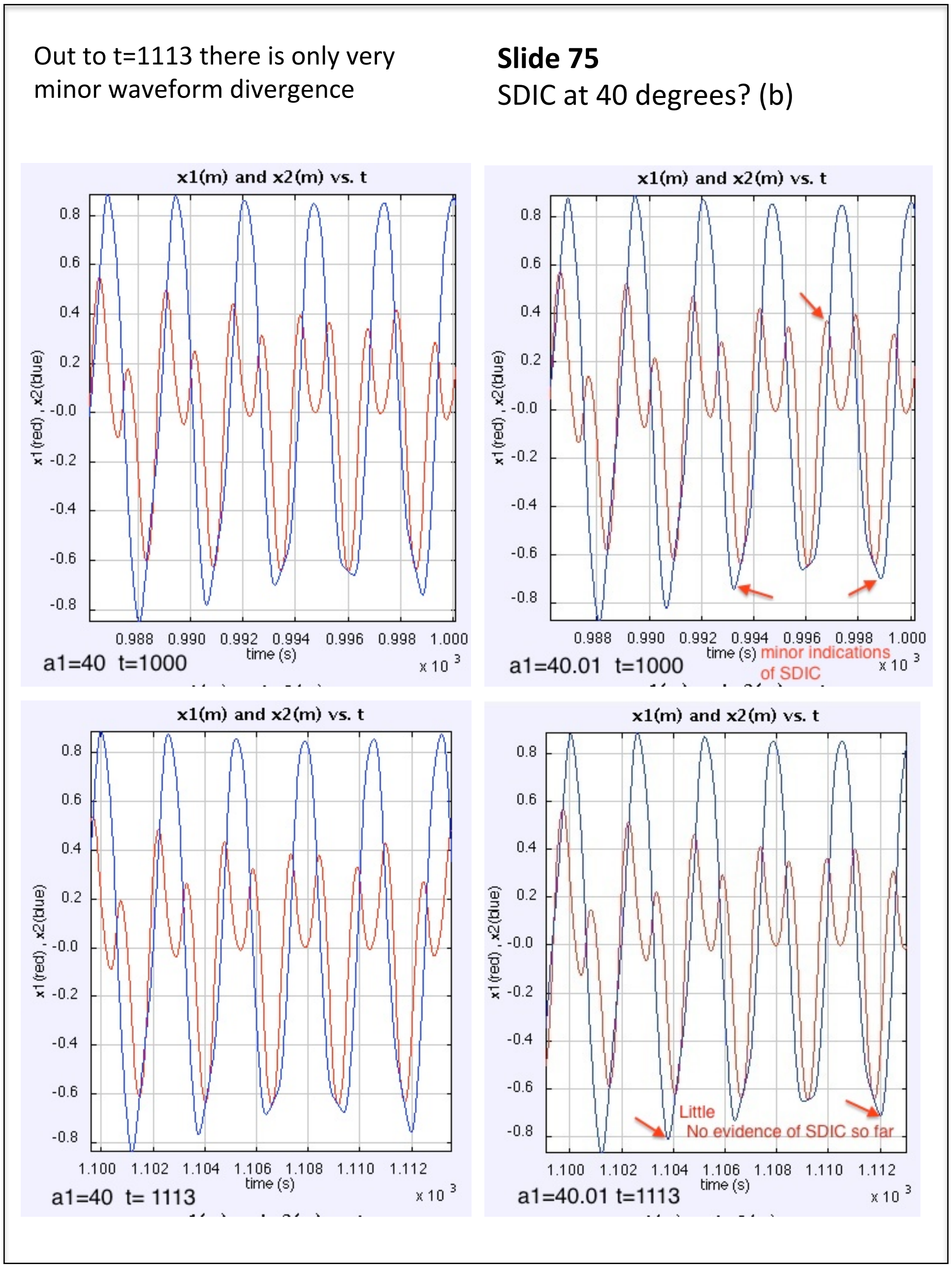
Minor waveform divergence or SDIC appears at or before t=488

Slide 74

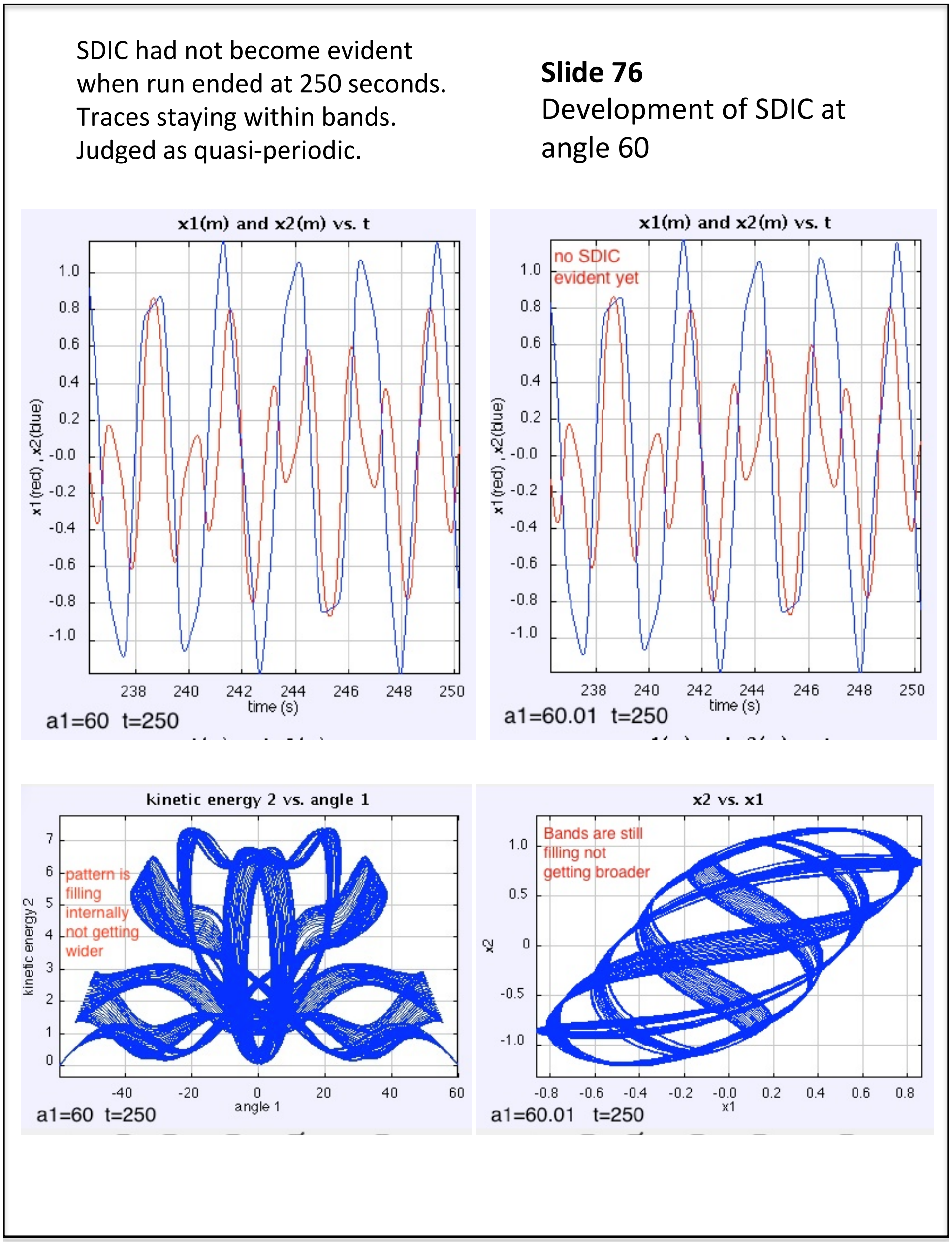
SDIC at 40 degrees? (a)



A2=0, m1=m2=l1=l2=1



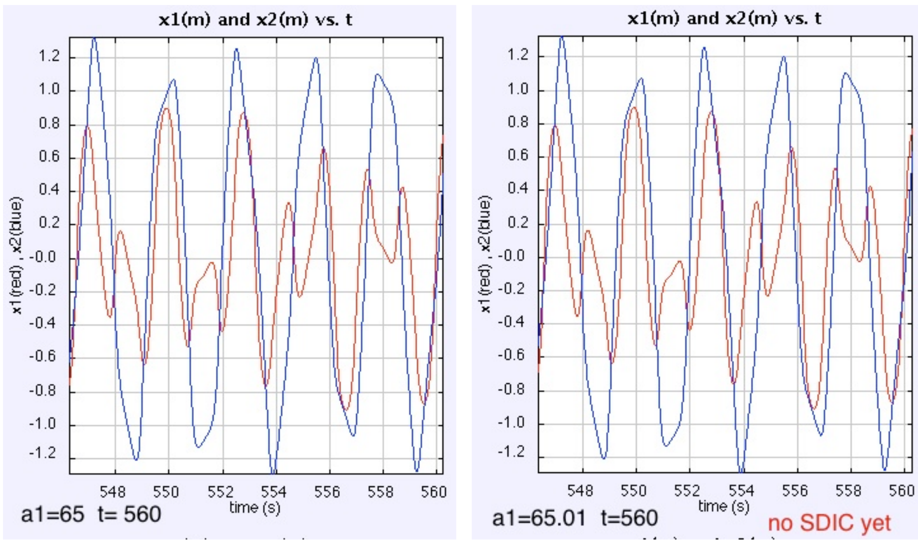
60 degrees: Note how the trace stayed within a band during the time it was plotted. In watching the trace make these patterns it did not appear to be expanding the width of the bands but rather infilling them. That might change in the long term.



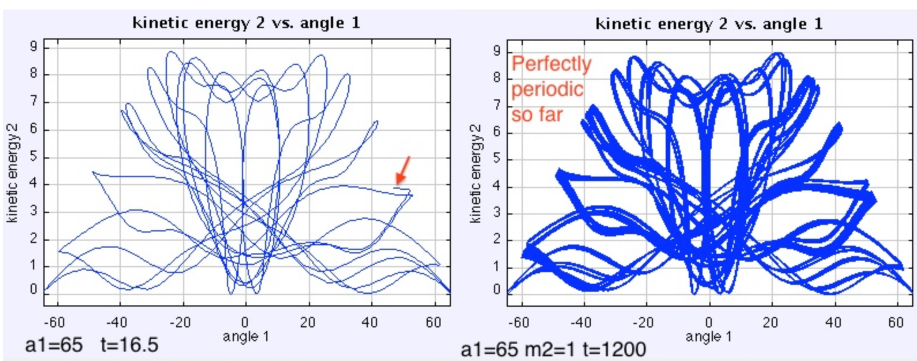
Angle 65: This long run showed no evidence of SDIC. It also turned out to be nearly perfectly periodic. In this case the bands had developed by t=421 but did not widen thereafter. This suggests that these bands do not necessarily continue to widen over time. That in contrast to other runs where they do widen and eventually fill the entire envelope. In an ffort to better understand chaos in the double pendulum it would be helpful to have a long series of ke2/a1 partial phase space diagrams like these made from fairly long runs at slightly increasing energy levels. They might be made into a movie. It would show how the bands constrict as the system approaches perfect periodicy, and where they grow to fill the entire envelope.

Slide 106
Development of SDIC
at angle 65

Stayed perfectly periodic with no SDIC so far, thus hasn't become chaotic.



Period was about 14 seconds



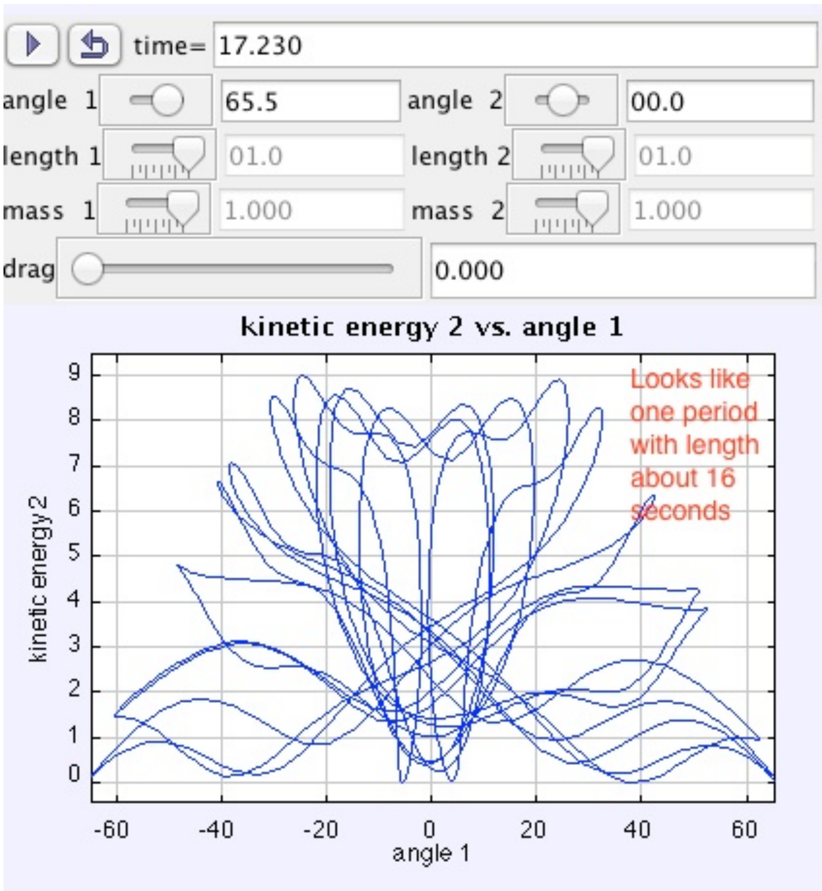
A2=0 m1=m2=l1=l2=1

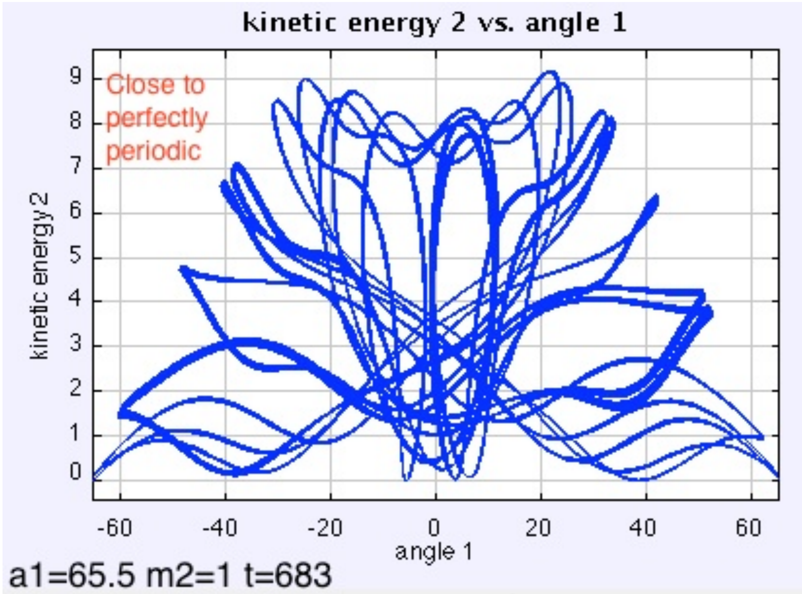
There was no change in width of bands from t=421 to t=1200

Angle 65.5: Retroactively I've added the results of a run at 65.5 degrees. It turned out to be almost perfectly periodic, since the trace stayed within a narrow band during a long 683-second run.

NOTE THIS: As shown below the system became chaotic when a1 was raised only one half degree more to 66 degrees. Going from perfectly periodic to chaotic with only a very small increase in a1 -and thus in energy- is worthy of note and may be important in trying to understand the root cause of chaos.

We now have two configurations that were perfectly periodic just before they became chaotic. It would be very interesting to know if all double pendulum configurations behave this way.

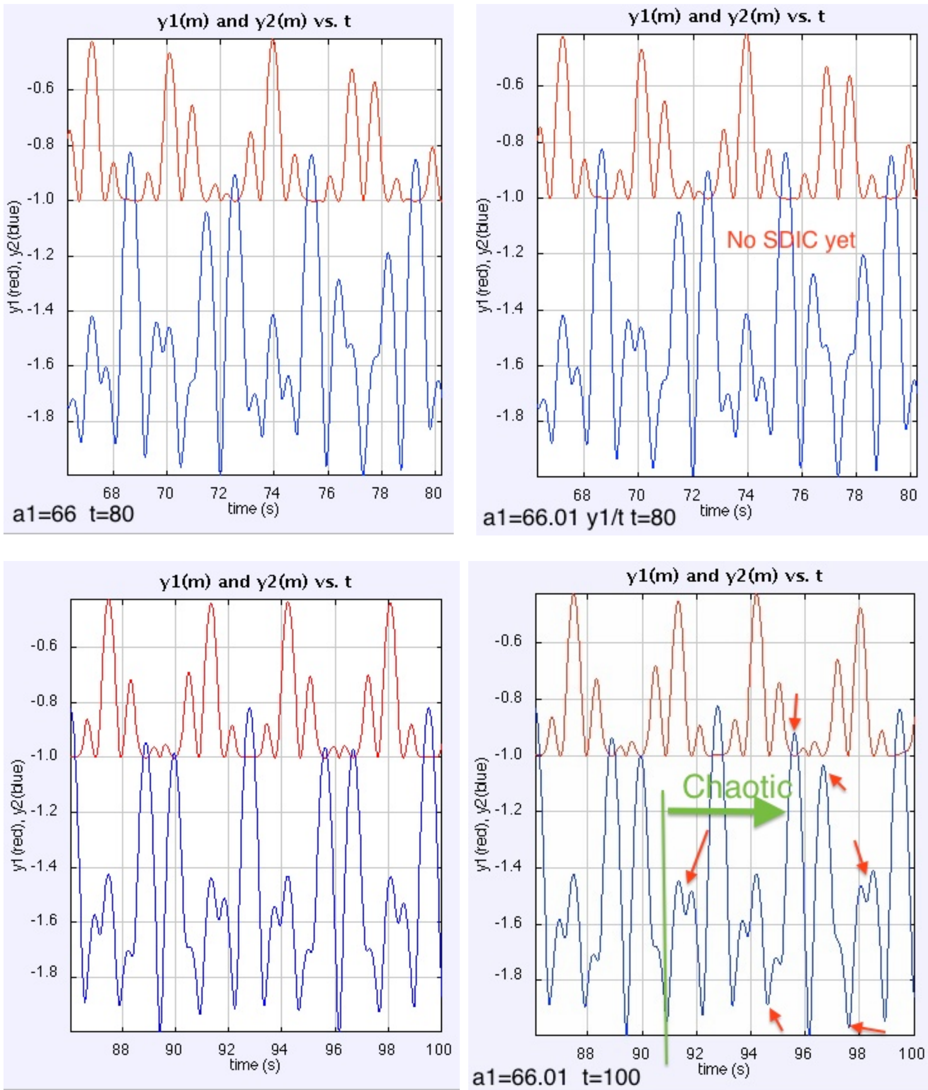




Angle 66: SDIC effects did not appear during the first 90 seconds, but became quite apparent after t=91. That makes 66 degrees the threshold to chaos for this particular set of bob masses, arm lengths and other initial conditions. With fine tuning the threshold may have been at a slightly lower angle like 65.7 degrees and if so it probably would have taken longer to manifest SDIC.

SDIC and thus chaos evident after t=91

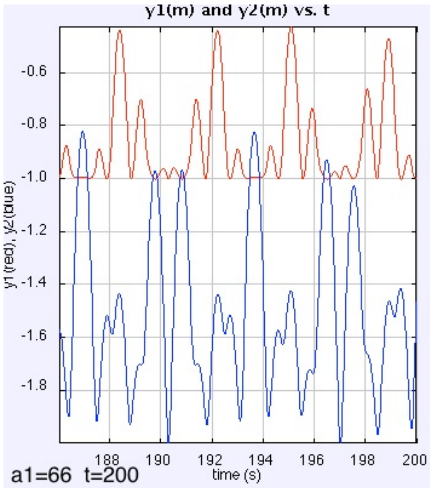
Slide 104
Development of SDIC at angle 66

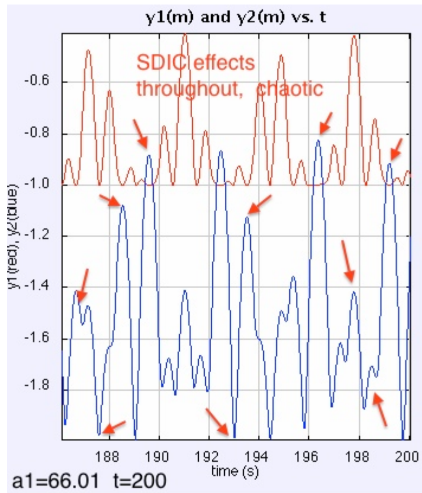


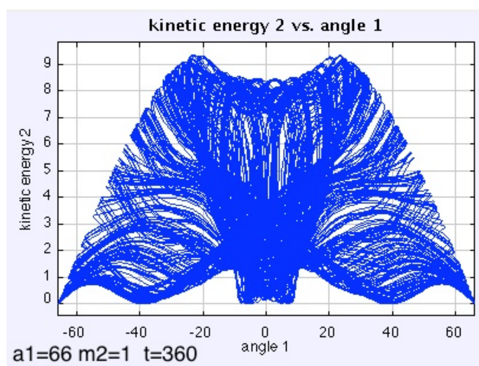
A1=0, m1=m2=l1=l2=1

Run had significant SDIC and thus was chaotic.

Slide 105
Development of SDIC at angle 66



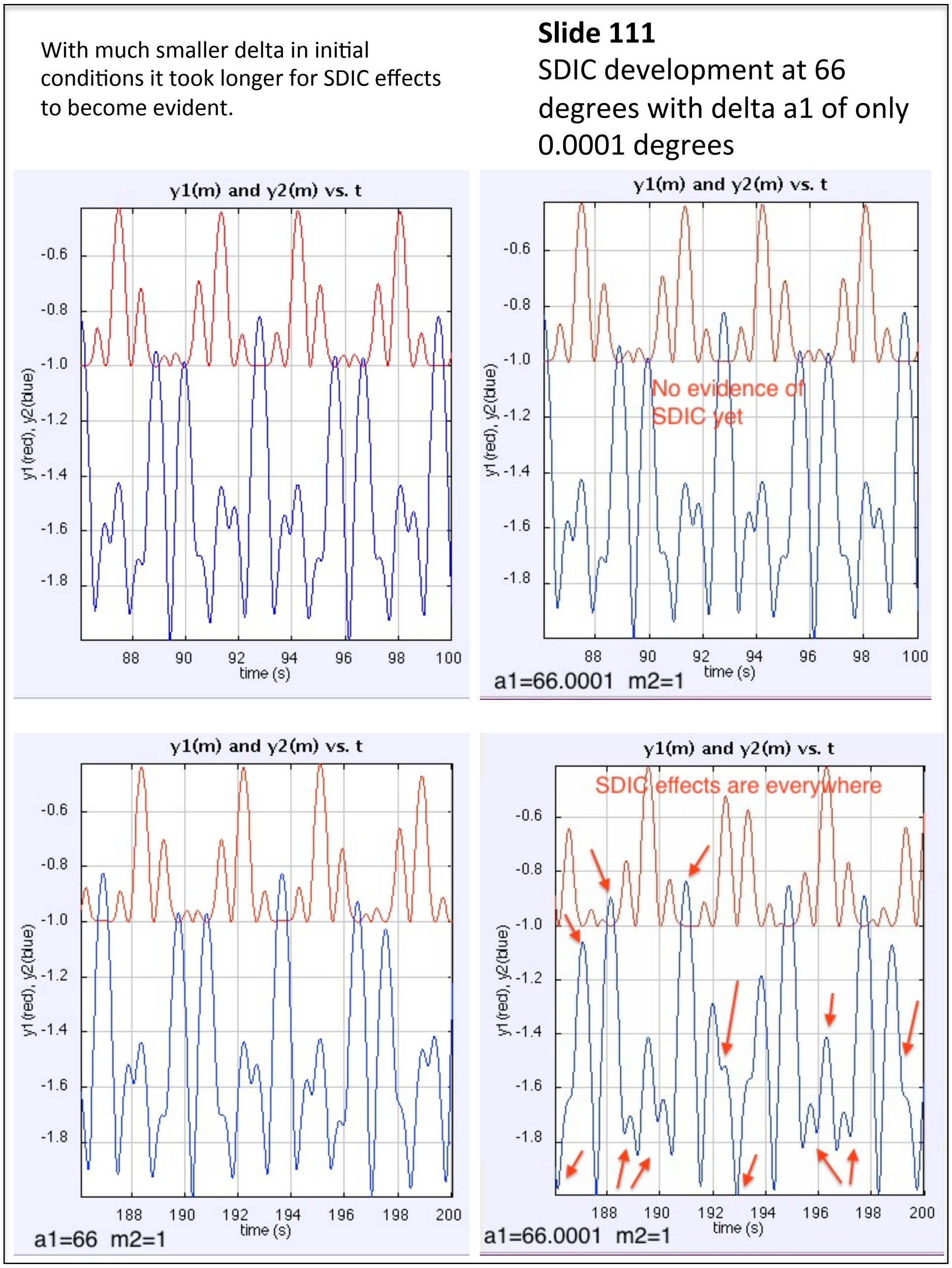




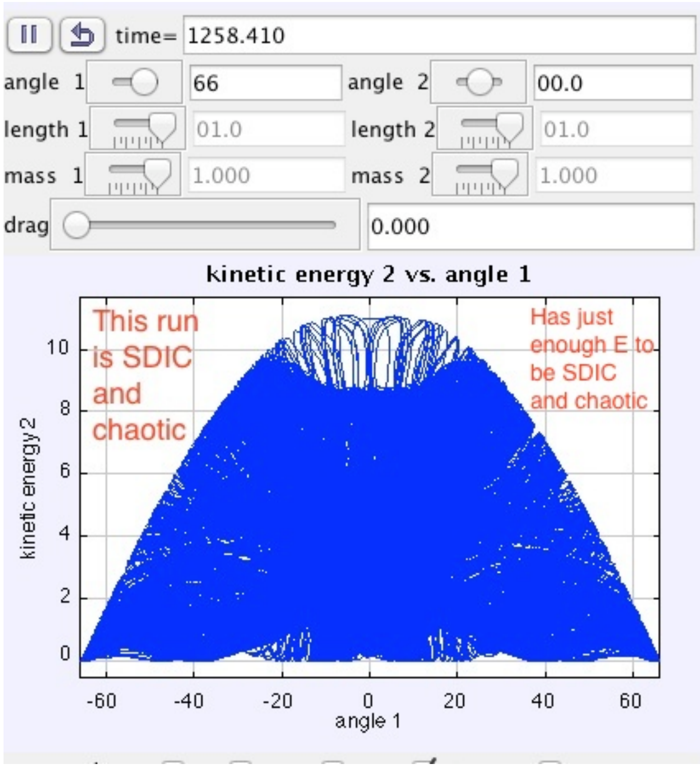
A2=0 m1=m2=l1=l2=1

The time it takes for the waveforms to diverge, in other words for the effects of SDIC to manifest, increases as the delta in initial conditions becomes smaller. This seems intuitively obvious. With a delta in a1 of 0.01 degrees it took 91 seconds. With a delta of 0.0001 degrees it took somewhere between 100 and 200 seconds. Slide 111 has the relevant screenshots.

With a difference in initial conditions of only 0.0001 degrees this is obviously a good example of <u>sensitive</u> dependence on initial conditions.



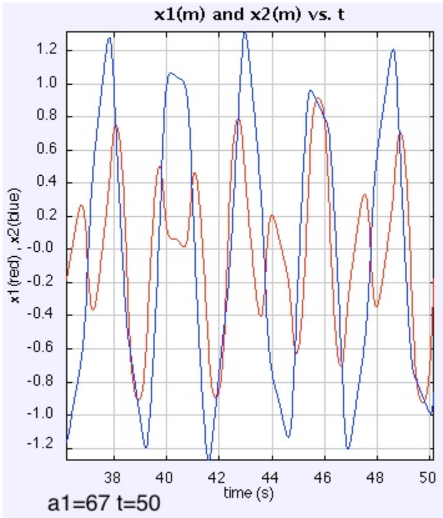
Here is the long-term plot of ke2/a1: It shows that chaotic operation ends up filling the entire envelope with traces because each cycle is different from those before. Its evident from the sparse traces at top that peak or spike values of ke2 are rare.

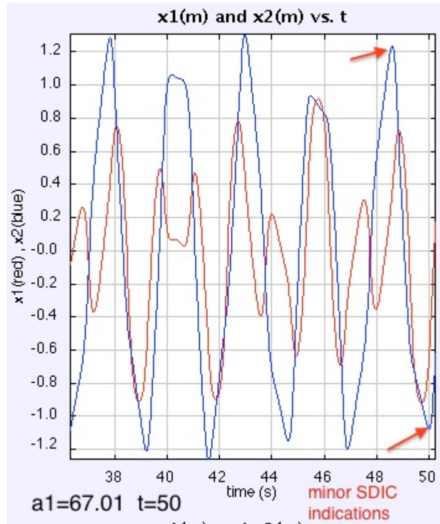


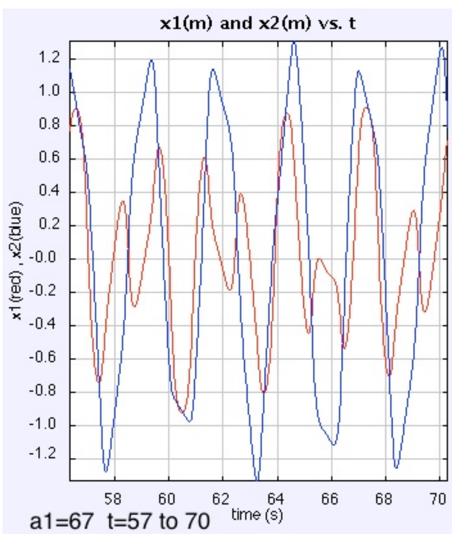
67 degrees: This time it took less time for SDIC effects to manifest, namely 71 seconds. Differences in wave shape and magnitude can be seen after that.

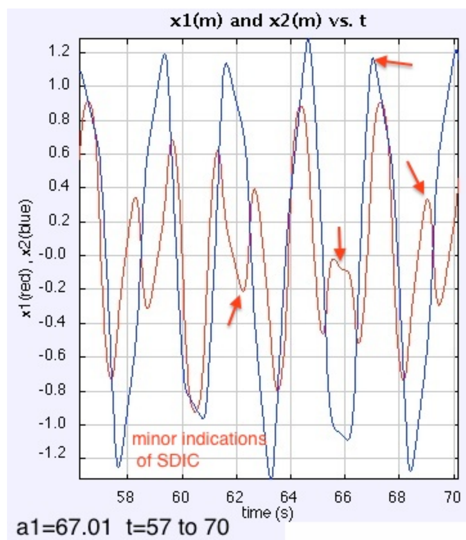
Slide 102
Development of SDIC at angle 67, part 1

Very little SDIC evident so far

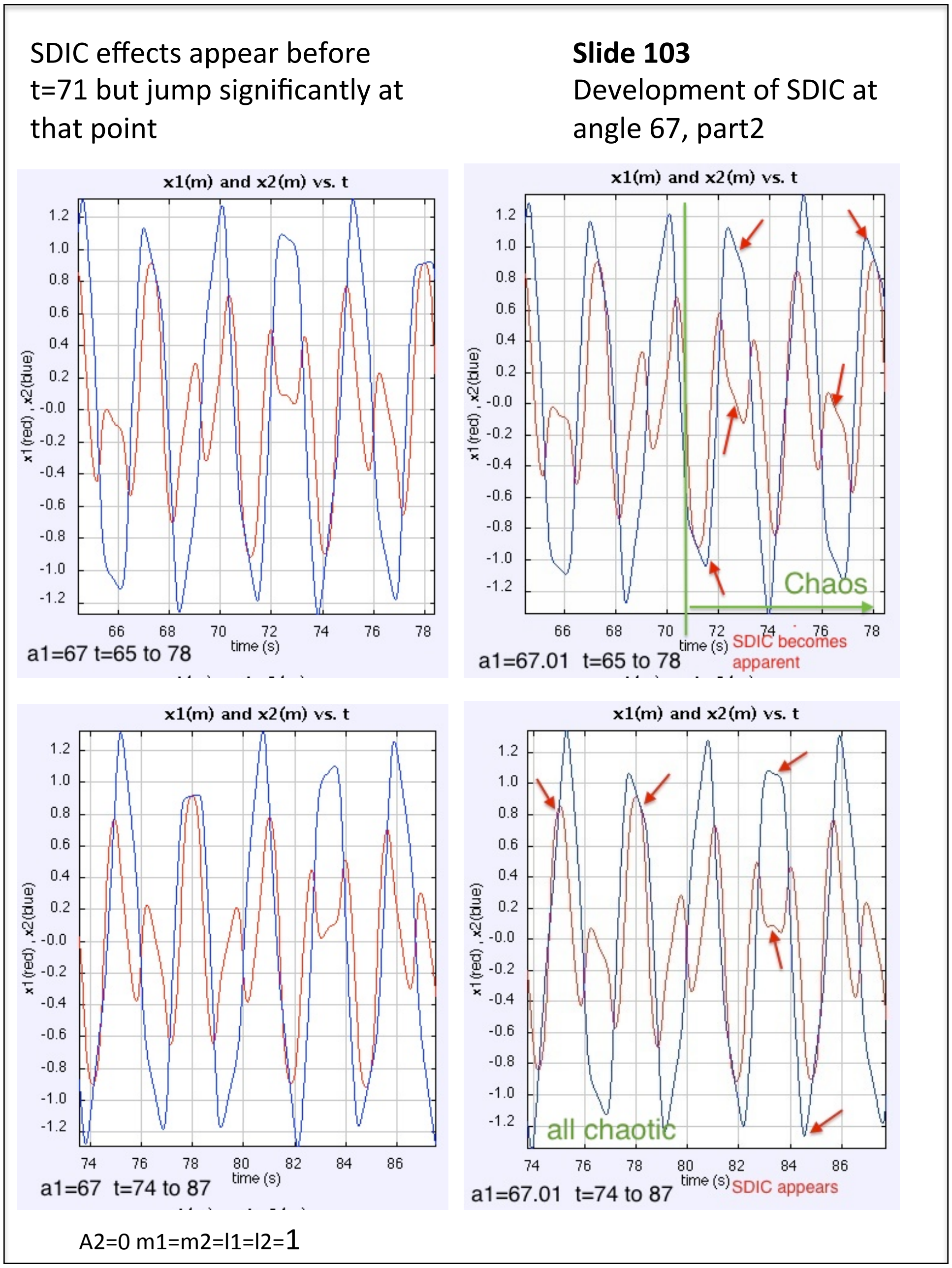




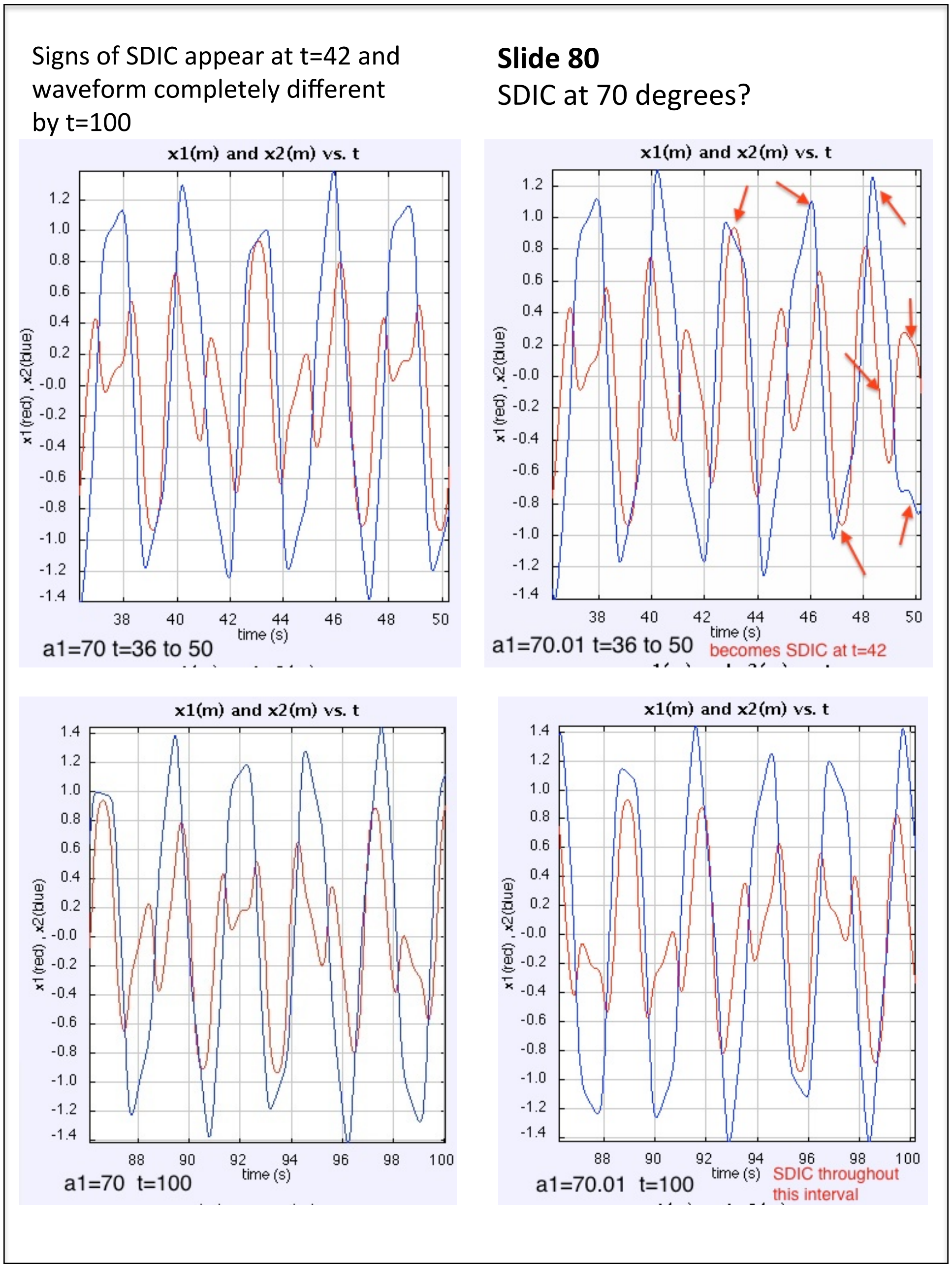




With a2=0, m1=m2=l1=l2=1



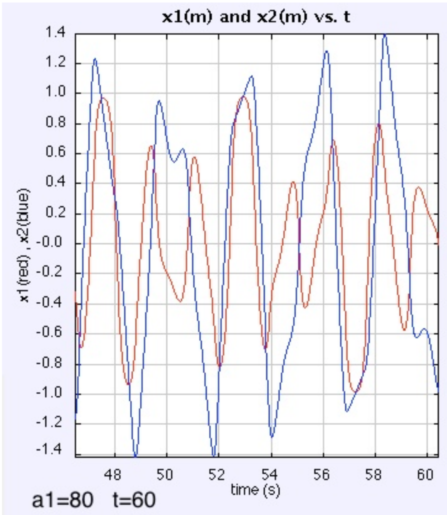
70 degrees: SDIC at about 42 seconds.

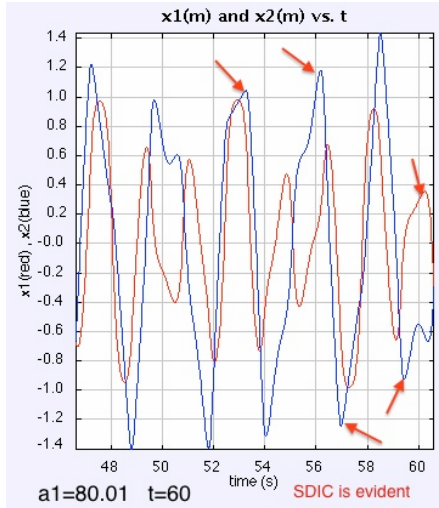


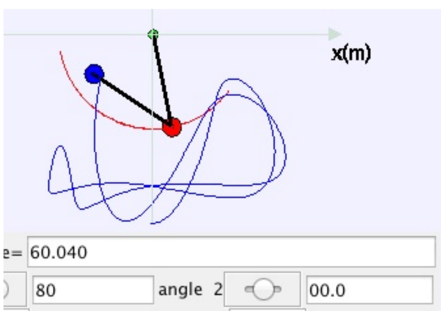
80 degrees: Differences can be seen in the actual bob paths. SDIC is noted at about 53 seconds. This is out of line with the declining trend so far. It may or may not be significant.

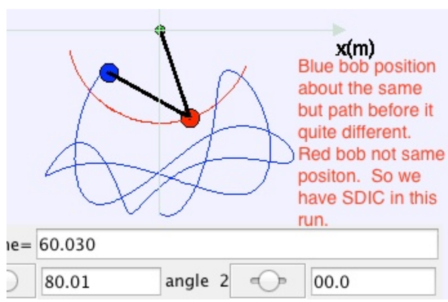
SDIC appears at t=53 and bob path completely different at t=60

Slide81
SDIC at 80 degrees?





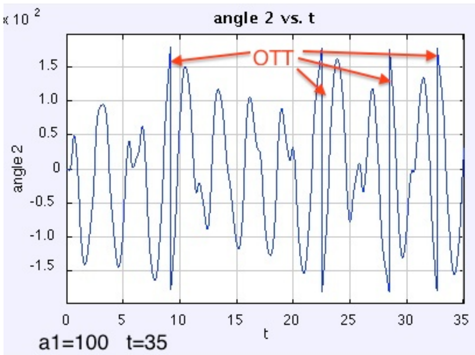


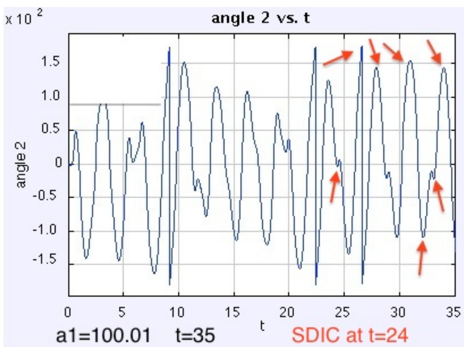


100 degrees: The messy ke2/a1 plot is typical of a chaotic double pendulum plot. There is however some element of pattern in these plots as one watches them form. Perhaps its like watching the Lorenz strange attractor in action. It too has a pattern that repeats in a general way, namely spiraling around in one wing until it crosses the center line and jumps to the other wing.

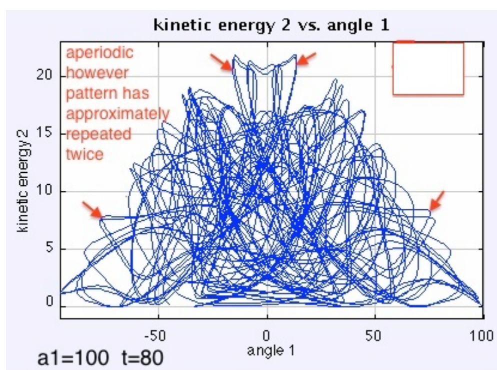
Slide 82
SDIC at 100 degrees

SDIC first appears at t=24. Note the difference in OTT events thereafter.





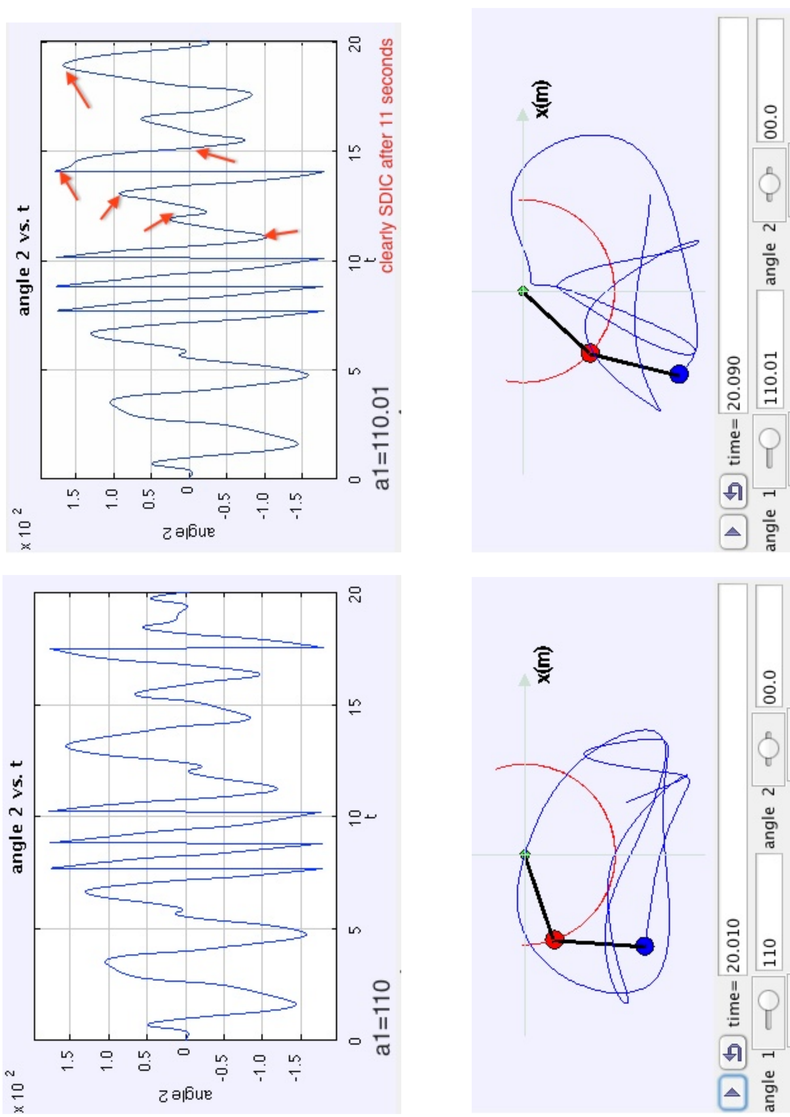
KE plot inserted for interest. Hard to ascribe meaning to it. Although chaotic at 100 degrees there is evidence of approximate pattern repetition over this first 80 seconds.



110 degrees:

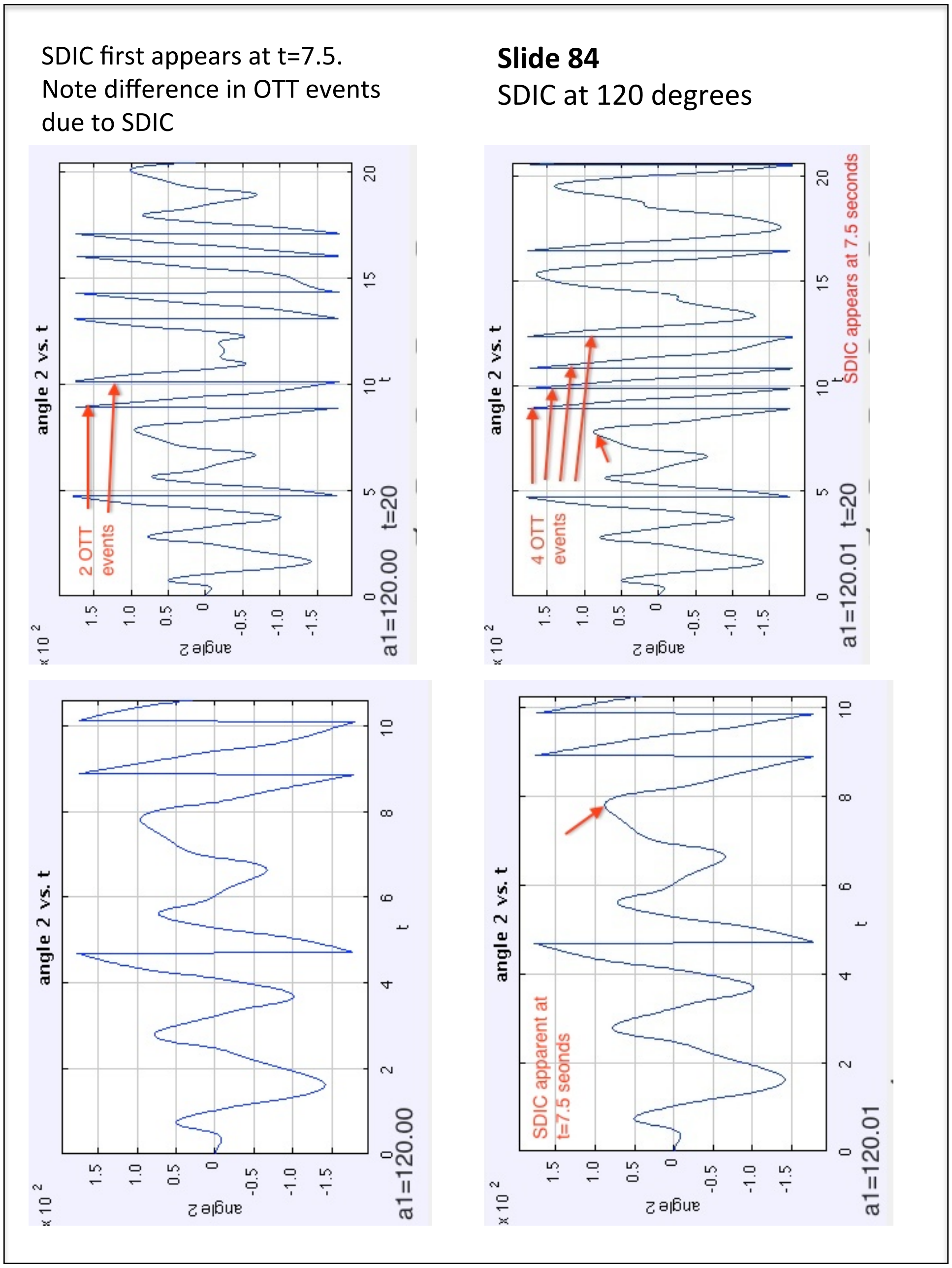
SDIC appears at t= 11.

Slide 83
SDIC at 110 degrees



Path of bob much different before 20 second screenshot

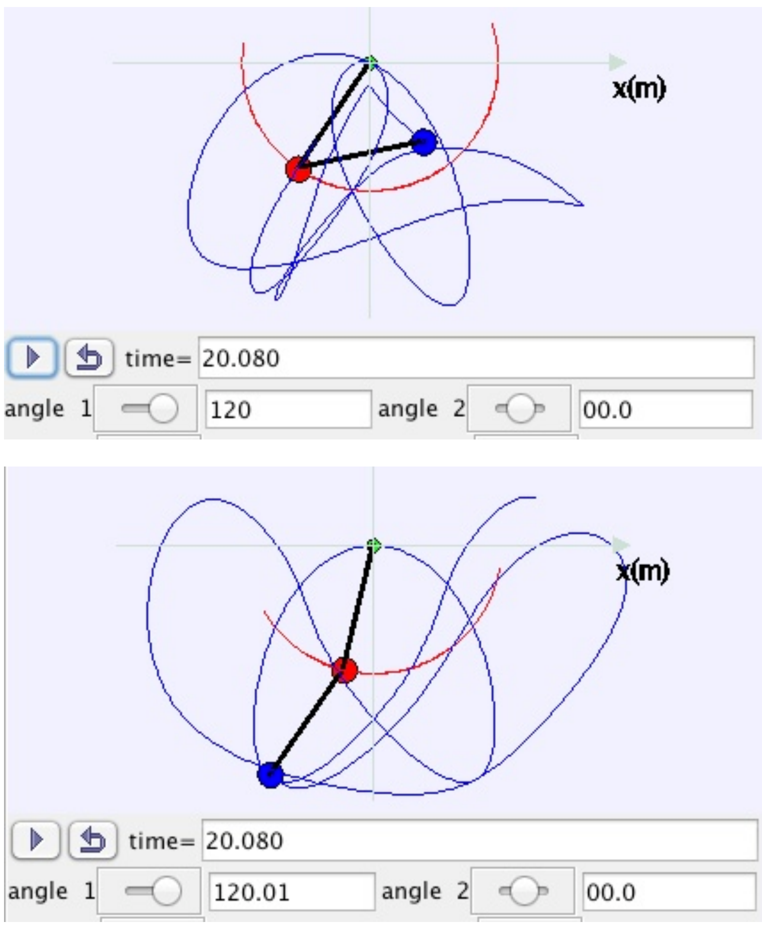
120 degrees:



This contrasts the bob paths during the first 20 seconds and is graphic illustration of how SDIC affects the actual physical behavior of the system.

Slide 85
Effect of SDIC on arm positions in 120 degree runs

Example of how SDIC makes prediction of exact future situation impossible, even over this short 20 second run. Both bob positions and paths are much different.



9.5 Impact of becoming chaotic

Perhaps one of the more important reasons to study chaos theory it to learn what the consequences when a historically periodic or quasi-periodic system becomes chaotic as a result of adding energy to it. The notion that global warming might cause some important environmental or ecological system to become chaotic makes this important to understand.

However to put this in perspective we need to know two things first. Is the system now periodic or quasi-periodic and if so how close is it to becoming chaotic. How close to crossing the threshold? We can probably answer both for the solar system, but that isn't a concern I've read about, possibly because the system changes so slowly.

Some earth systems like global temperature are apparently quasi-periodic on timeframes of thousands of years due to variations in solar radiation, earth sun distance, and earths tilt. I set them aside partly since they are not things we can effect.

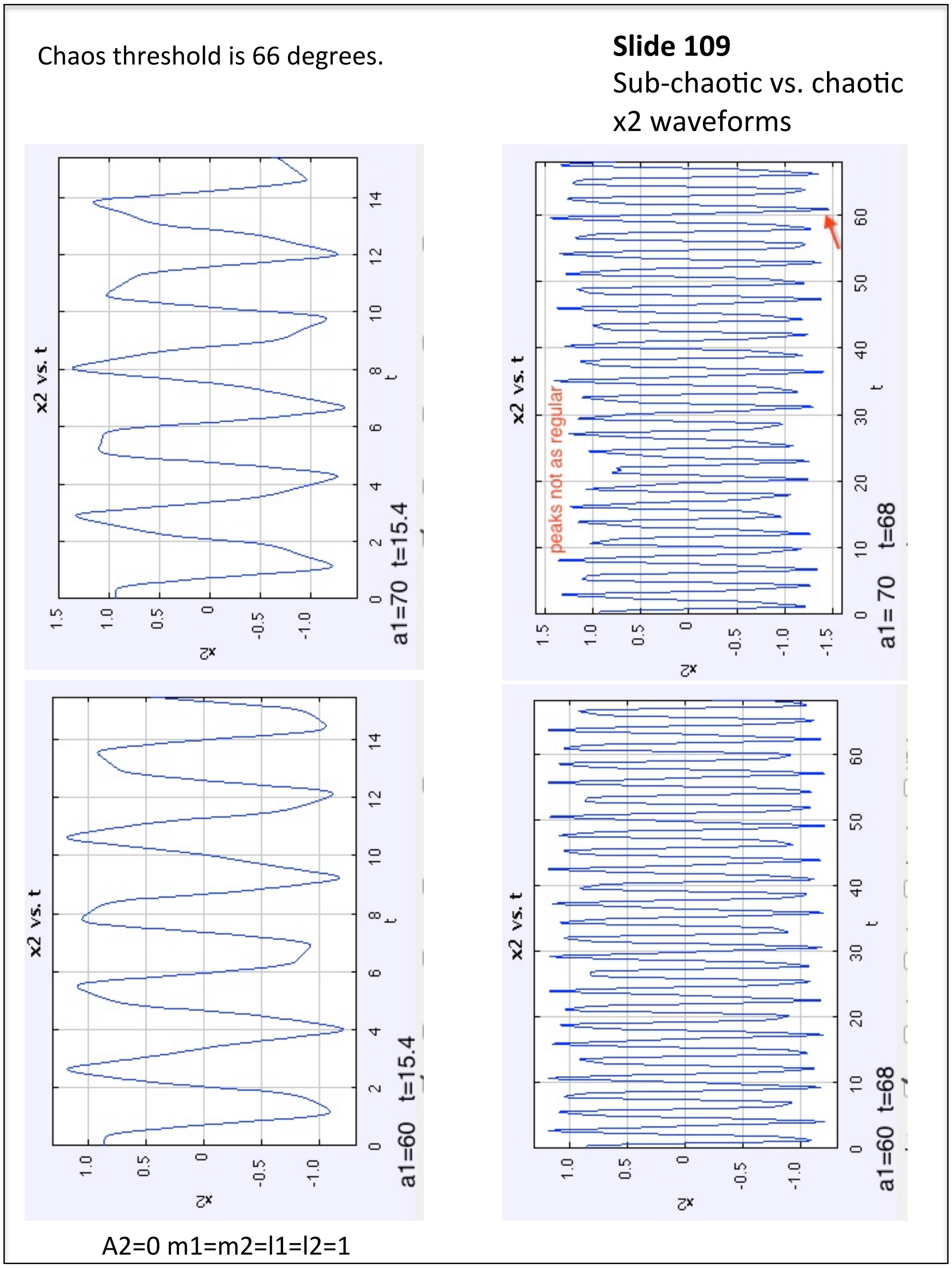
Probably the largest, most massive system humanity can and has affected is climate by virtue of greenhouse gas emissions, which are slowly raising average global temperatures. I've not investigated whether global temperature is oscillating because its just one variable in a system that is internally oscillating like a spring-mass system, an N-body system, or a double-pendulum. The carbon cycle could be one such system. Alternately global temperature may simply be responding to the variation in solar radiation caused by exogenous forces as listed above. Investigating these thoughts is beyond the present scope.

I must note that there has been much research into the occasionally chaotic behavior associated with heart rhythms and aerodynamic flutter. And some other systems as well.

It isn't easy to apply anything we learn from studying the transition into chaos of simple systems like the double pendulum to more complex systems. With that in mind what follows is a brief look at the former. My approach was first to compare behavior of the double pendulum when it was sub-chaotic versus when it was chaotic. Would there be any notable differences in the way the system oscillated? The second was an attempt to model the system as it transitioned from sub-chaotic to chaotic and see if anything dramatic happened at that point.

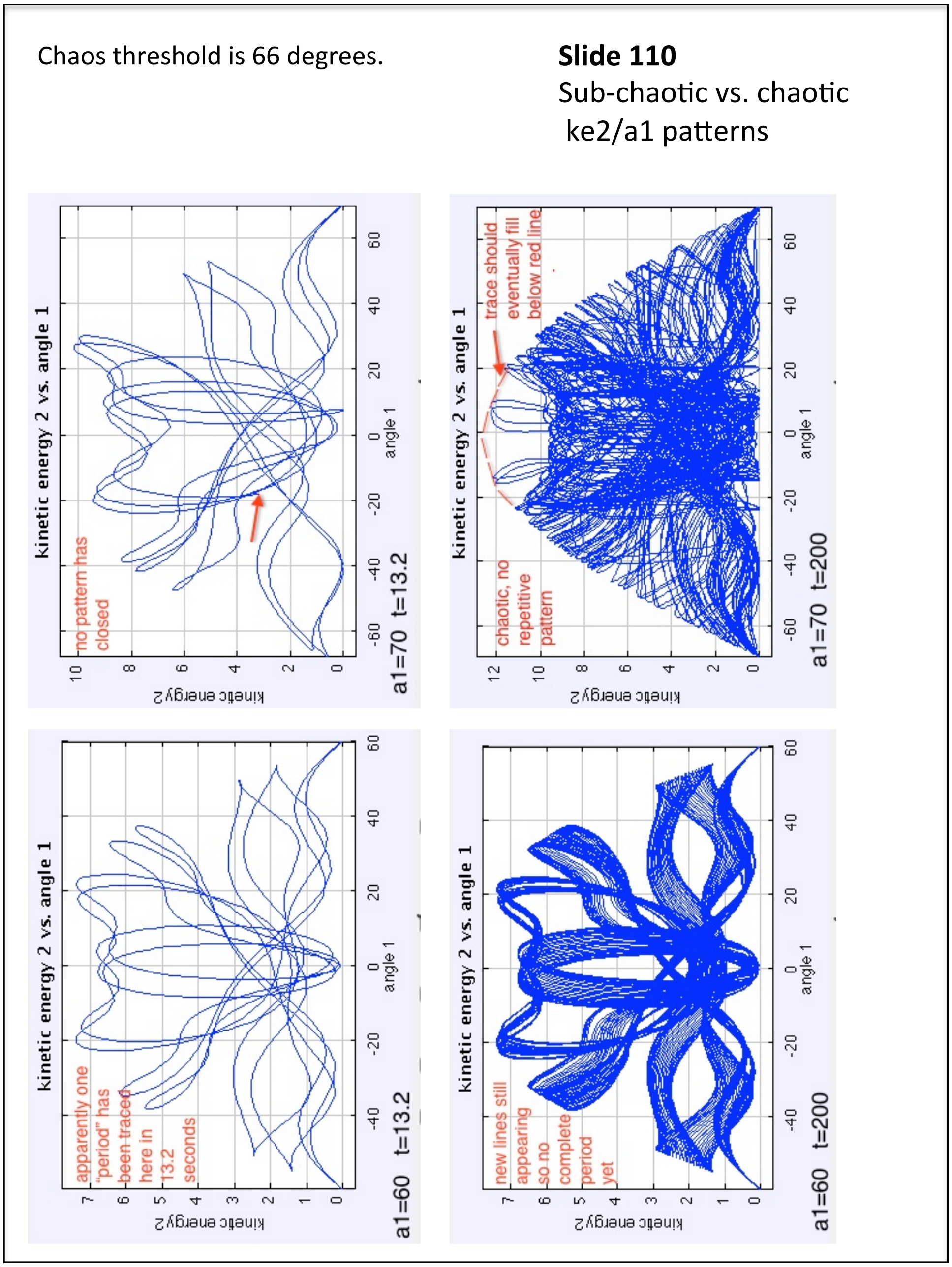
None of this produced any dramatic results but its worth presenting anyway. I still feel its an important question and hope others will investigate it thoroughly.

Slide 109 compares waveforms of the double pendulum when it was sub-chaotic and quasi-periodic in a run with a1 set to 60 degrees to a chaotic run made with a1 at 70 degrees. The short-term view showing just a few cycles shows the behavior is virtually identical. The system oscillates at the same frequency, the waves are no more sharp or rounded. Their heights vary about the same amount from one cycle to the next. This suggests, not proves, that the transition into chaos would hardly be noticed in the short term by anyone depending on this system.

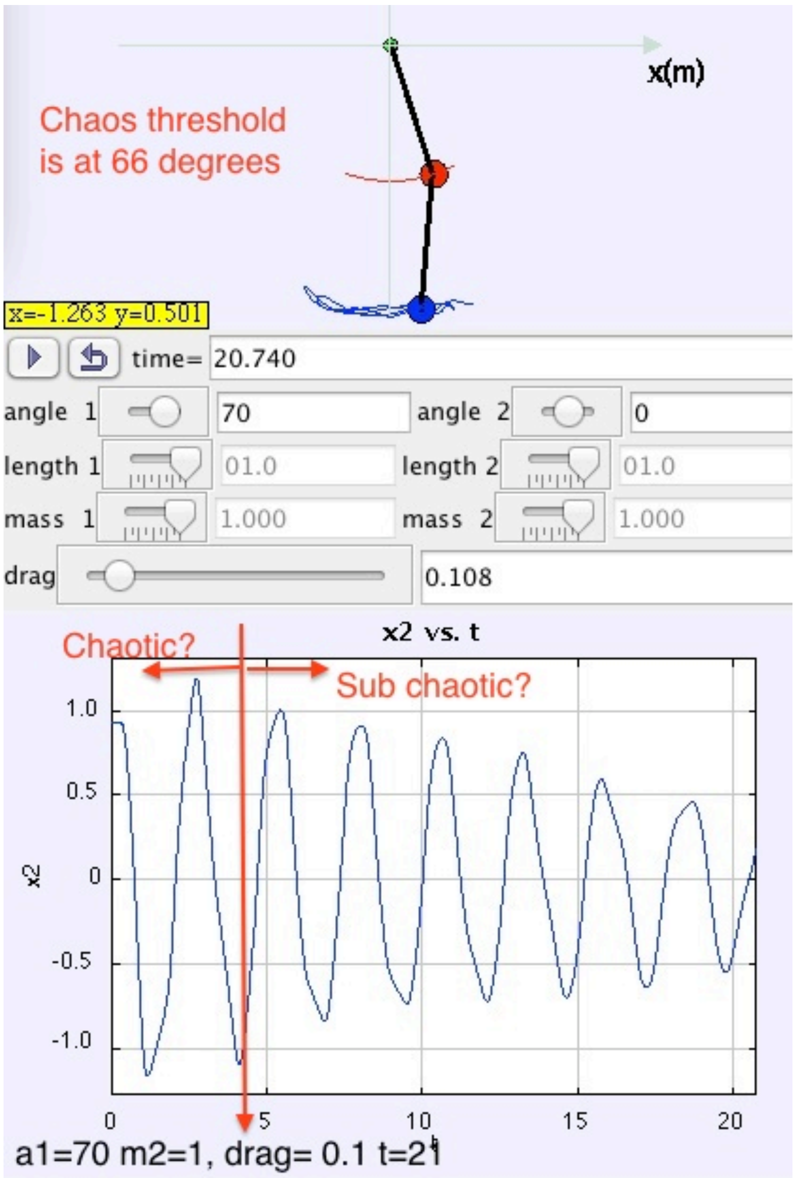


At first glance the longer-term behavior doesn't seem radically different either. The peaks and valleys grow a bit in magnitude but that's because the level of energy has changed by a significant percent. The frequency remains the same. The heights of the peaks seem to vary about the same percent. However close inspection shows that the 60 degree run has a repetitive pattern which produces two high peaks separated by two lower ones, then two highs separated by one low and so forth. In contrast the 70 degree run is much less consistent because its chaotic.

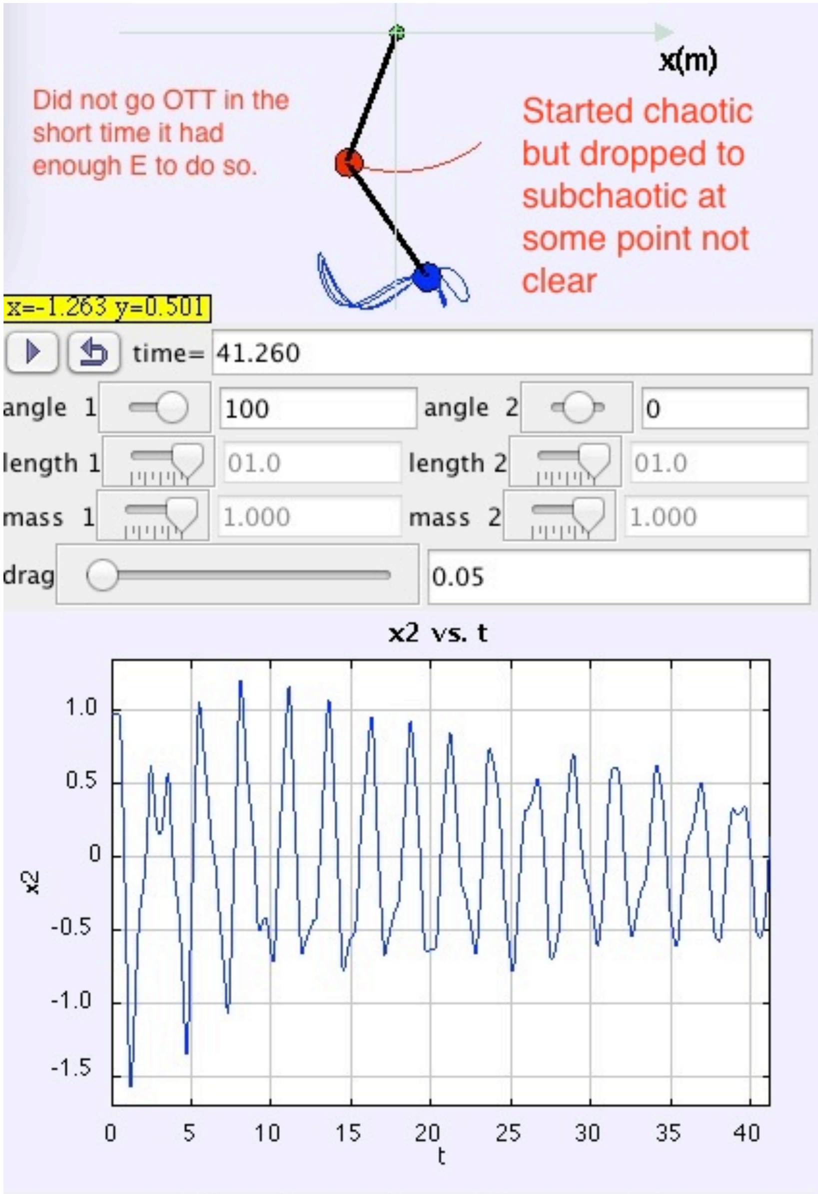
Slide 110 conveys much the same overall message. The short-term pattern of behavior is much the same. In the long term the values change over a wider range.



Next it was decided to try simulating a system transitioning thru the chaos barrier. The model didn't support the gradual addition of energy but it did support its gradual diminution by allowing frictional drag to be included. It was assumed that the transition from sub-chaotic to chaotic would be a mirror image of the transition from chaos to sub-chaos. We know that the system was chaotic when started at 70 degrees and since the overall wave height dropped to less than it was in the 60 run it should have been sub-chaotic at the end. Just were the transition occurred is uncertain. I simply marked my best guess. IF this represents reality then nothing dramatic would happen if the double pendulum had enough energy added to make it chaotic.   Viewing the real pendulum gives further support to this finding. Its movement is fluid as it slows from wildly chaotic to a stop.



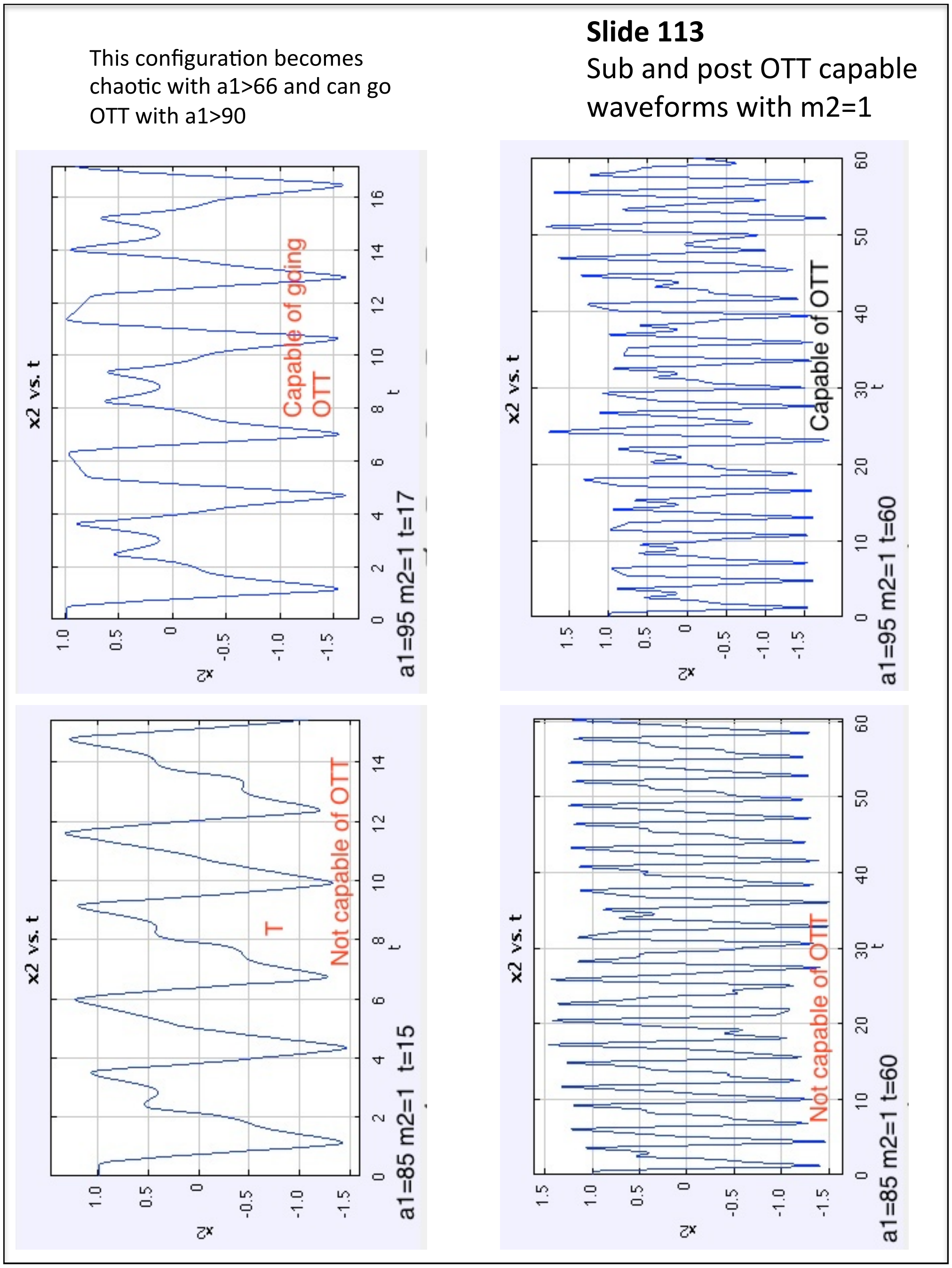
The following run started from a still higher energy. It transitioned the threshold without any noticeable impact on the waveform.



IF these results apply to real-world systems it suggests there would be no drastic and immediate change in behavior if they were driven into chaos by the addition of energy. I cannot reconcile this opinion with the drastic change that occurs when a regularly beating heart becomes arrhythmic. Perhaps that's an entirely different situation.

9.6 Impact of becoming capable of OTT events

Slide 113 shows the results of examining the waveforms above and below the energy level needed to cause occasional OTT events, as was also done in section 8.2. Again no significant differences were observed.



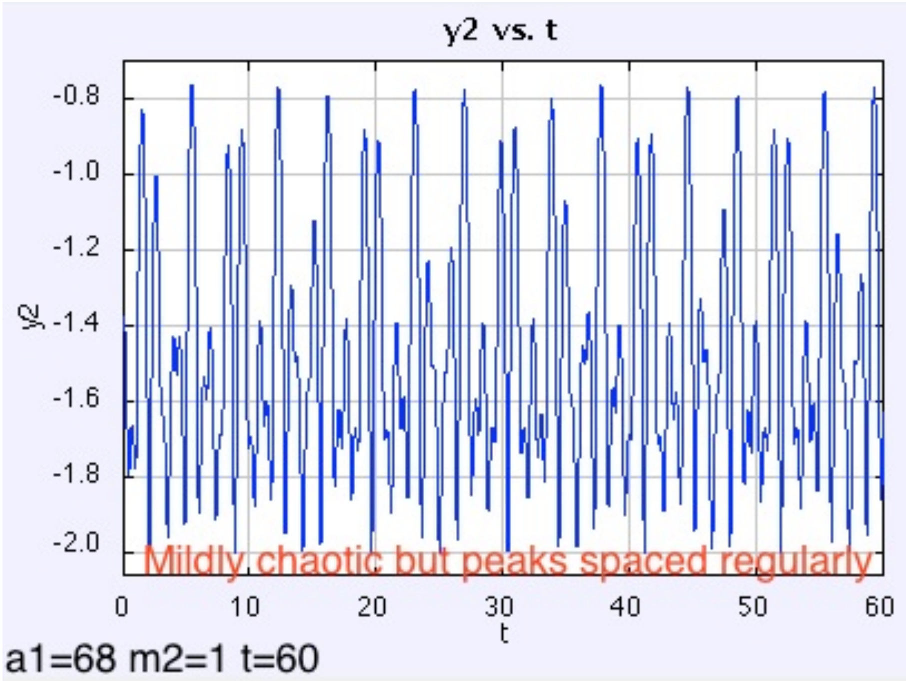
9.7 Are calms and spikes generic to chaos?

One of the most significant aspects of chaotic oscillation is the way in which a variable can oscillate gently for some random length of time and then suddenly spike to a much higher value. Its analogous to the 500-year storm, the prolonged draught ending with a flood, or a major stock market crash.

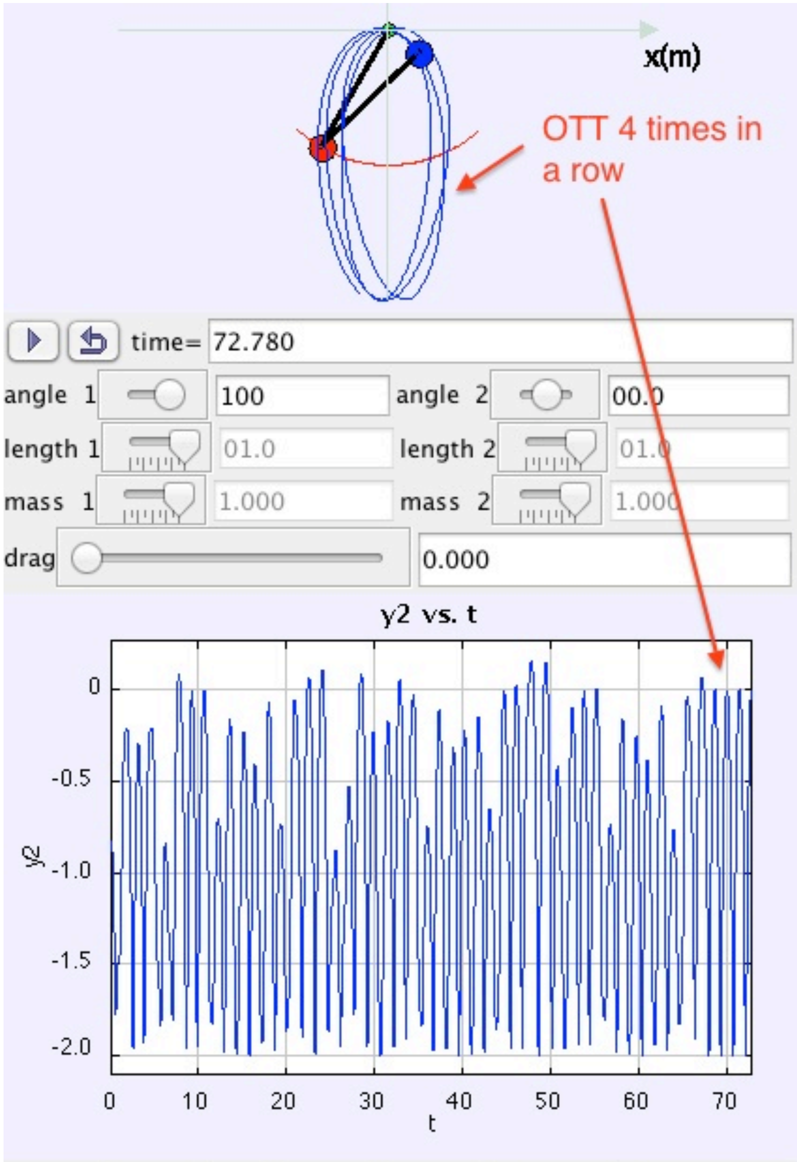
I've noticed such behavior in a number or runs and emphasized this type behavior before. Nevertheless, like it seems like everything else about double pendulum behavior I'm not certain that calms of random lengths interrupted by one or more spikes is always the case when the system is chaotic. Thus as I near fishing this book I've conducted a few more runs to better explore this question. These are presented below.

This screenshot is from a 68-dress run just over the threshold with barely enough energy to exhibit SDIC and thus be classified as chaotic. Obviously there are no long calm periods and the spikes occur on a fairly regular basis. In this particular case if a system gained enough energy to cross the threshold into chaos one might not notice much of a change in behavior, in other words we wouldn't suddenly be faced by random length calm periods followed by spikes. That's important because our greatest concern might be to know if something like global warming might tip some natural system into chaos what the effect might be.

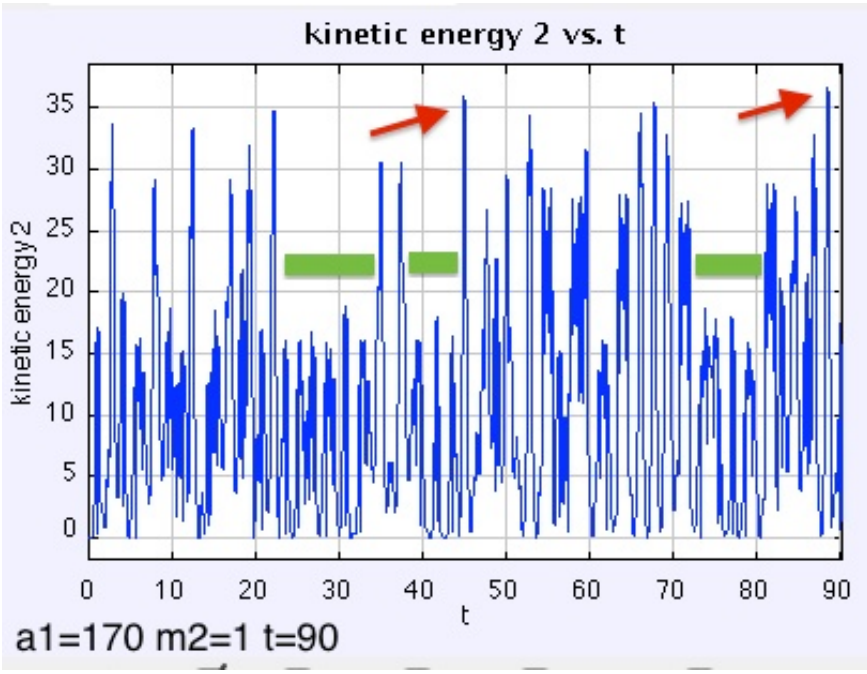
It also suggests that a defining aspect of chaos may not be those calms and spikes, which I had previously though were generic to chaos. (This subject is full of complicating surprises, and hoping to simplify things with generalizations is dangerous.)



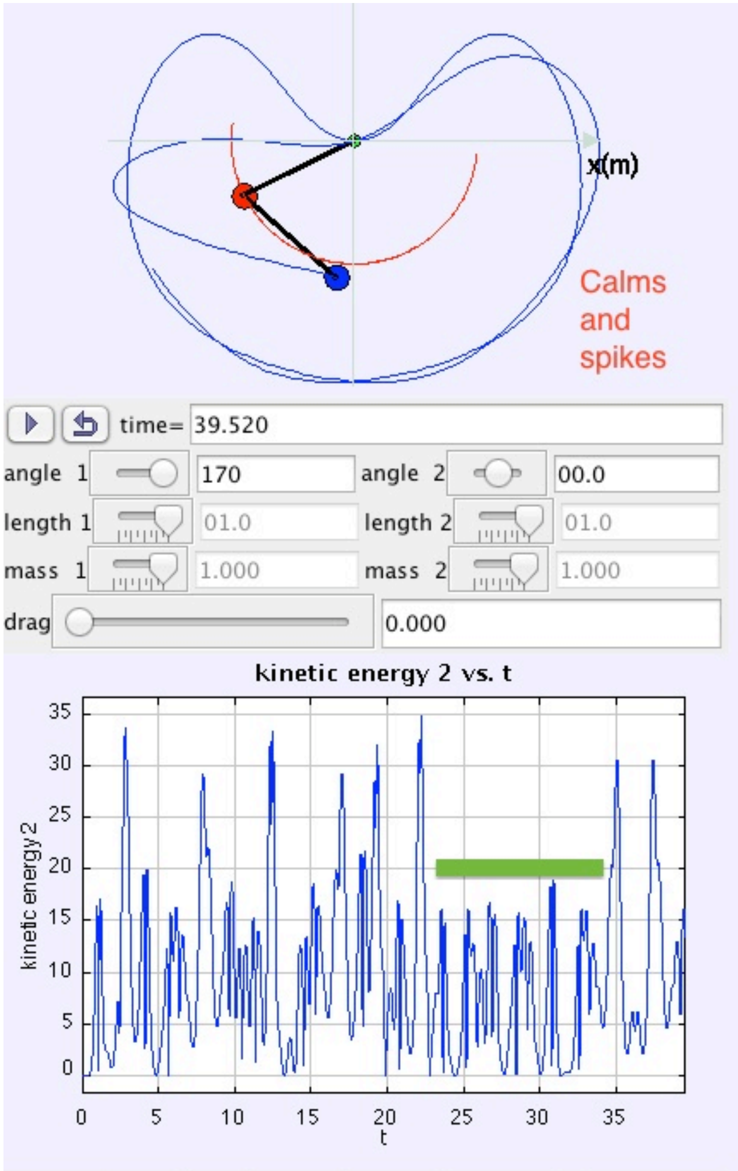
This run with a1=100 doesn't show and noticeable calm periods. The blue bob had just spun over the top four times when the screenshot was taken. Y2 only needs to reach zero for that to occur.



This run with a1=170 is strongly chaotic and now we do see random length calms highlighted by green bars. Notice the intense spikes that end the first two calms. Notice also that really high spikes occur relatively rarely. Contrast this waveform with the 68 degree one above. They are both chaotic but clearly different.



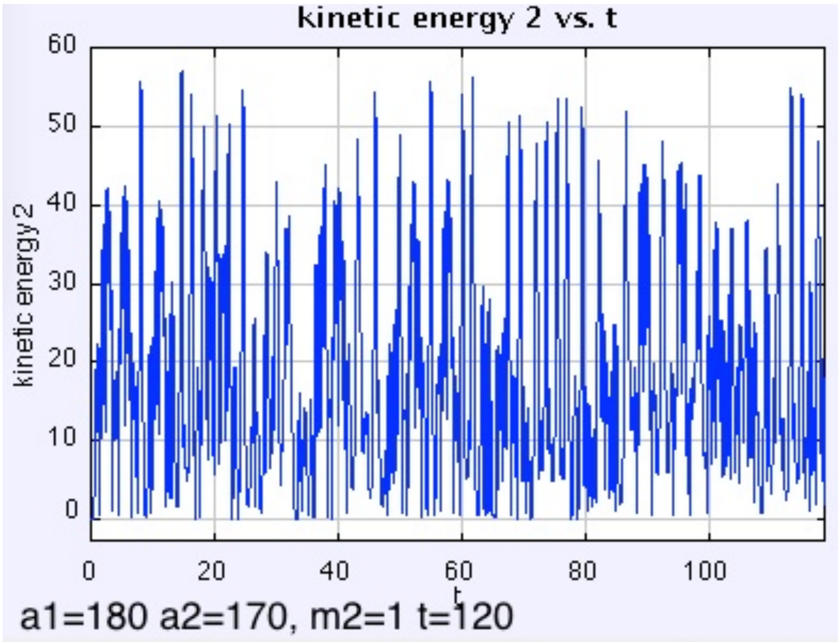
This is the first 40 seconds from that same run.



Its interesting to speculate about what this might mean IF it applied to a real-world system. For example assume some natural system like climate behaved this way. Perhaps the green represents a long period with relatively consistent rainfall, and we have grown to take that for granted. The economic system and folks in general have assumed it will continue forever. Then all of a sudden –and with no gradual increase to warn us- we suddenly get severe floods or draughts.

If this type of behavior does not apply to important real-world systems then this exercise in analyzing a toy system may have little practical value.

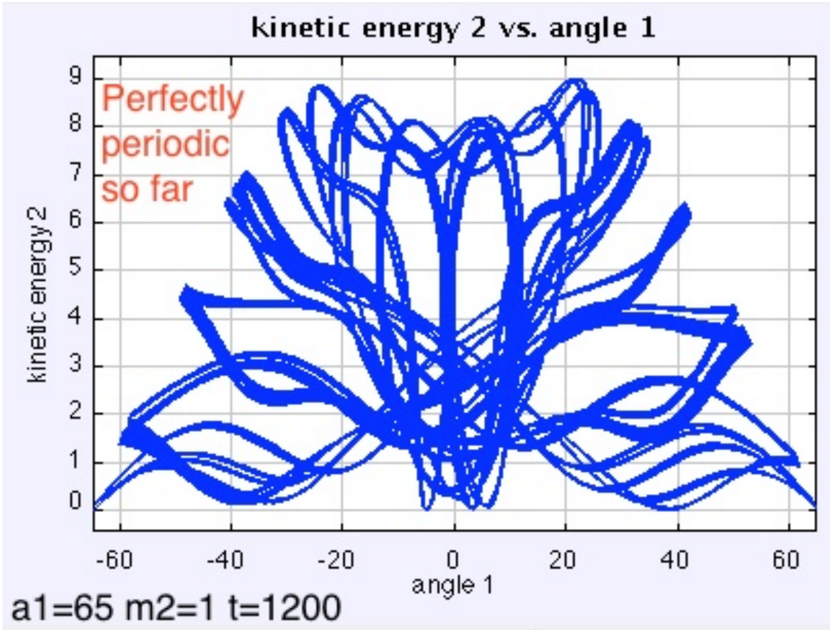
The waveform below was from an even higher energy run that was strongly chaotic as well. The random peaks and valleys we see is typical of chaotic operation. While there are somewhat quieter periods they don't seem as clearly demarked or dramatic as in the 170-degree run above.

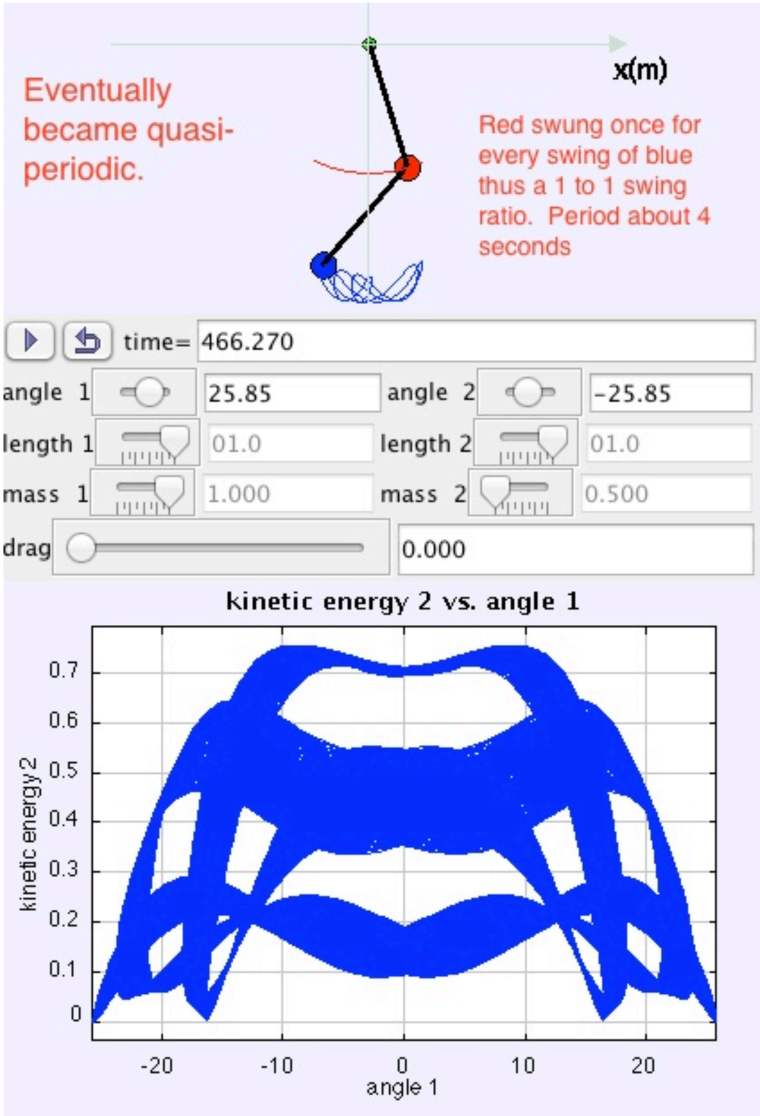


This set of runs show that chaos in the double pendulum does not always result in waveforms with noticeable calm periods interrupted at random by spikes. Further investigation is needed to better map this behavior and find out what conditions do produce waveforms with well-defined calms and spikes.

9.8 Is quasi-periodic behavior actually perfectly periodic in disguise?

This is another topic I just thought to address briefly when this section was supposedly finished. I've noticed in a number of runs that what I have sometimes called quasi-periodic and sometimes called perfectly periodic produce a ke2/a1 partial phase space plot where all the traces stay confined within bands. Two examples are as follows. Each run lasted long enough that if the trace was going to go outside the band it presumably should have done so in less time than these runs lasted.


kinetic energy 2 vs. angle 1
Perfectly
periodic
so far
kinetic energy 2
angle 1
9
8
7
6
5
4
3
2
1
0
-60
-40
-20
0
20
40
60
a1=65 m2=1 t=1200

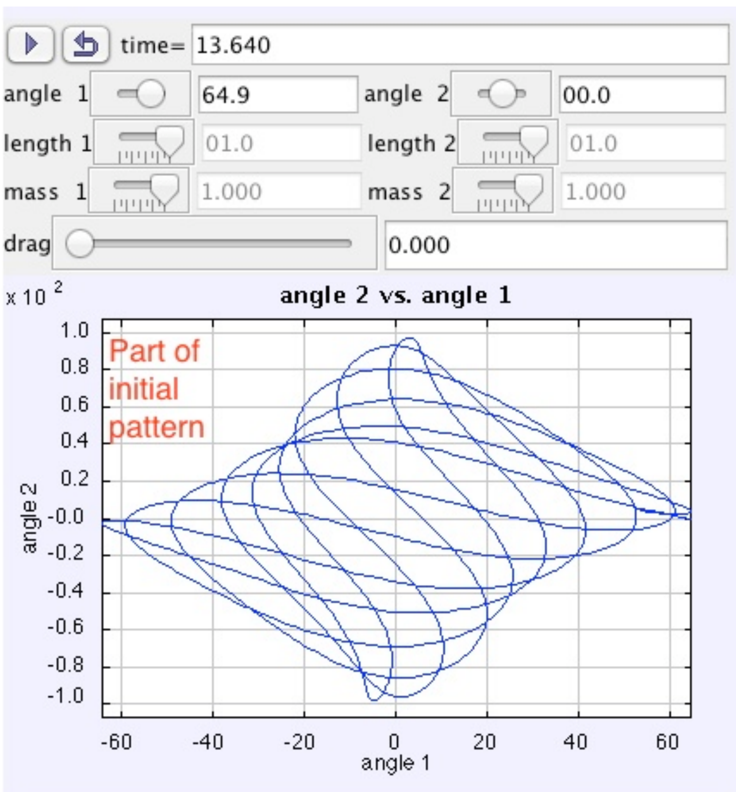


It's a subjective call but generally if the trace stayed within a narrow band I teneded to label the run as perfectly periodic. If the band was wide I called it quasi-periodic. Before now I hadn't giving much thought about what the traces might be doing within those bands.

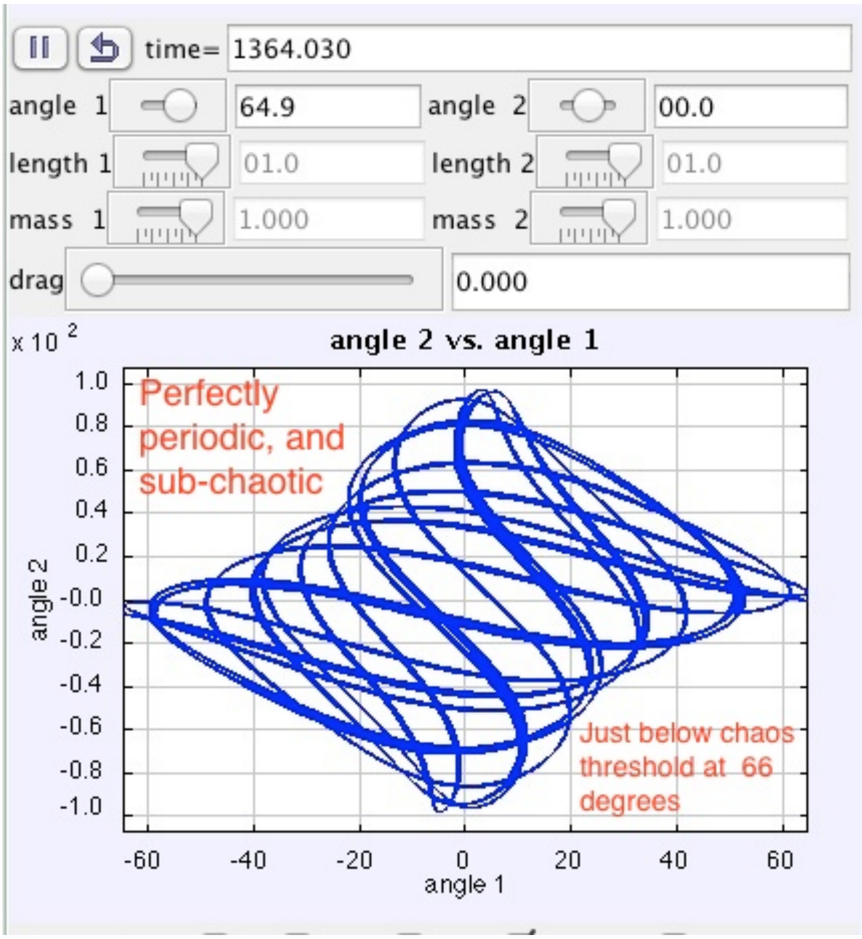
Now I think there are at least two possibilities. First, the trace may be wandering around chaotically within the band, possibly never tracing the same path twice. This could cause it to eventually fill in the entire band. Perhaps every new trace would fit between previous ones reminiscent of the way it apparently does in the strange attractor for the Lorenz equations. Perhaps this is a fractal structure.

However the alternative may be that the band is filled by a series of similar shaped but not identical patterns and that sequence of patterns does exactly repeat time after time. If so the behavior might actually be perfectly periodic behavior in disguise. The real period length would be the sum of the times needed to produce each pattern in that sequence.

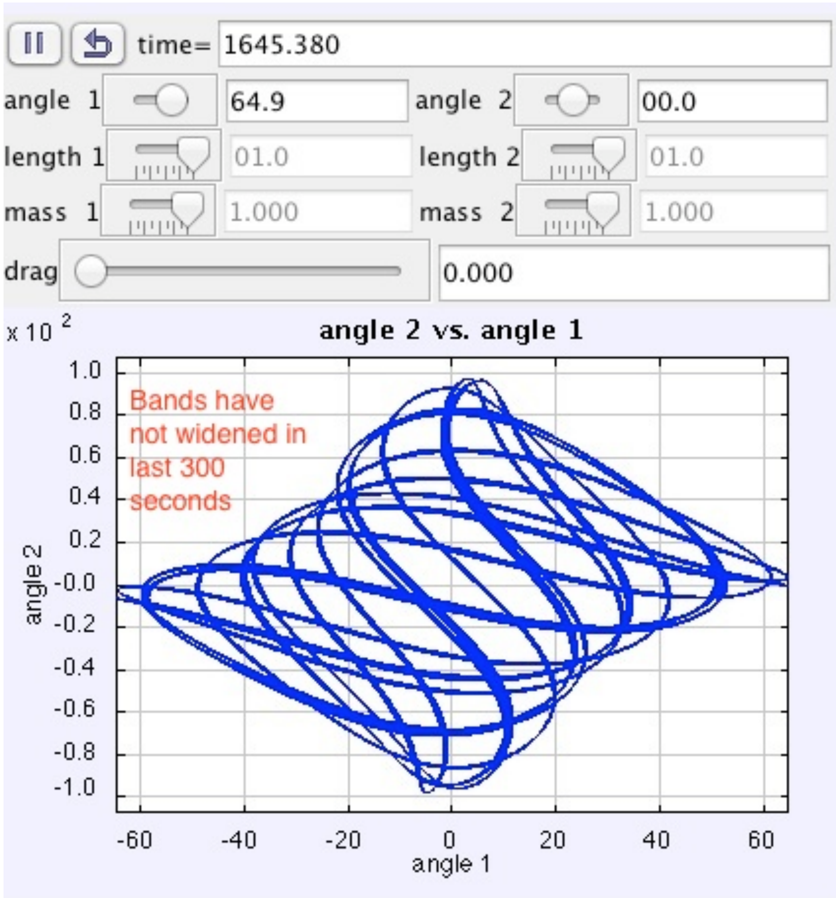
To explore this a bit consider the following three screenshots for a run I chose to call perfectly periodic. It was made at 64.9 degrees, slightly less than needed to make it chaotic. Despite labeling it perfectly periodic the pattern did not replicate over and over exactly so as to form a single line but at times the successive patterns were slightly different causing them to fall within a narrow band. The first screenshot shows the trace as it paints most the initial pattern. The "period" to complete this pattern was about 15 seconds.



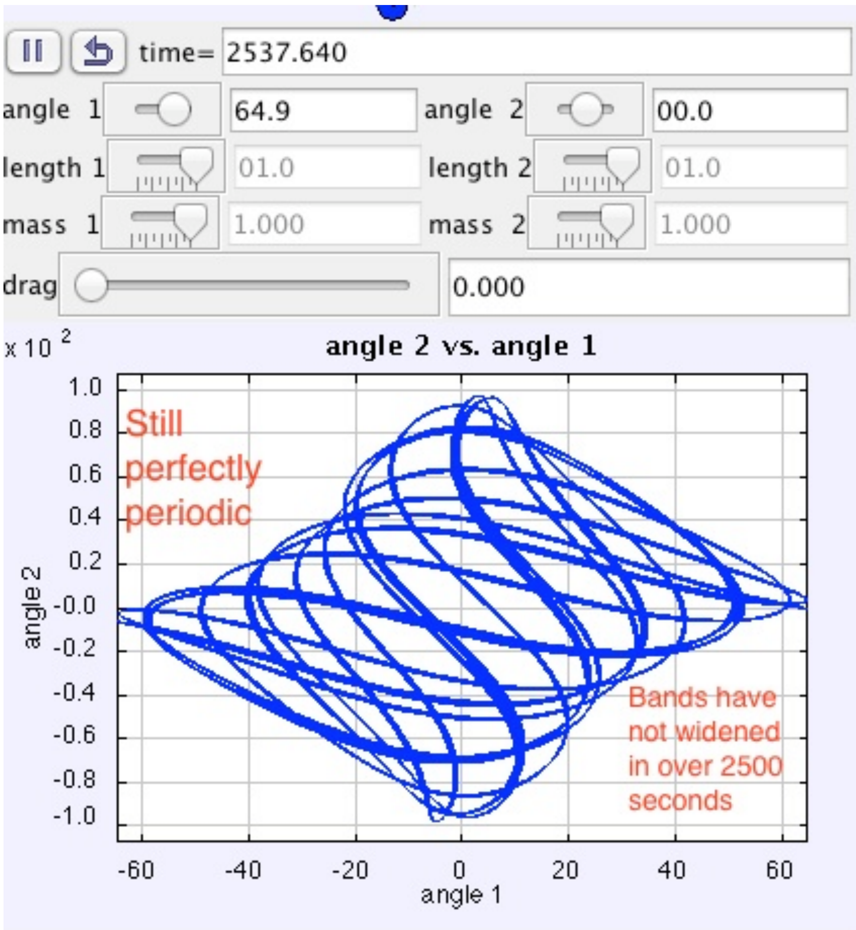
This shows those wider bands after 1364 seconds. That a long run and if the trace was going to widen the bands it should have had time to do so.



Nevertheless the run was continued. Here it is after 1645 seconds.

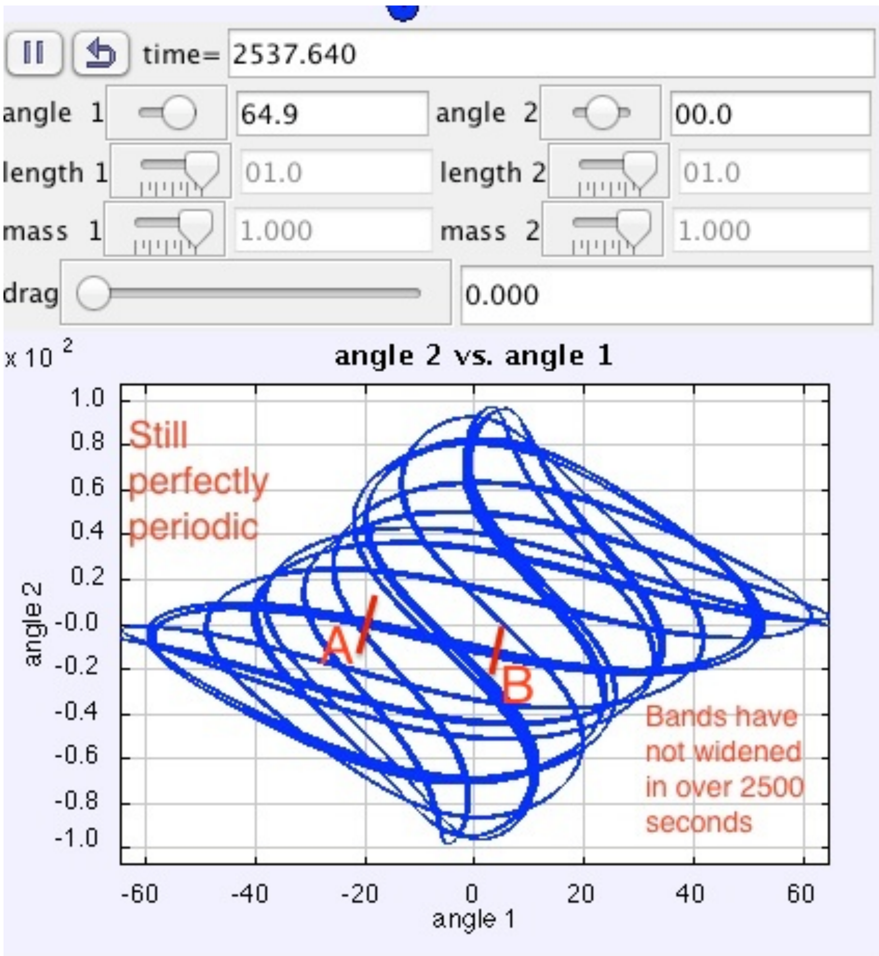


And finally after 2537 seconds.



It seem unlikely that allowing it run still longer would have widened these already fairly narrow bands any wider.

Now I propose a way to see if the wide bands are actually formed by a series of slightly different patterns that repeats time after time thus making this a perfectly periodic run, albeit with a period that is some multiple of the time it took to complete just one pattern, which was roughly 15 seconds.



Lets focus on the band between A and B in the screenshot above. Its of course very crude but the band between A and B appears about 4 times wider than where it narrows most. (Obviously the pixel size gets involved here). Thus it seems possible that what we are seeing is 4 slightly different patterns that lie almost atop each other. Where the band is widest they are slightly different. Where the band is narrowest they are virtually identical. The time needed to complete each pattern is about 15 seconds so if four of them overlap the real period needed to complete this pattern was 60 seconds. In other words this was perfectly periodic, not quasi-periodic, behavior with a period of 60 seconds. If it took 5 cycles to make this pattern the period was 75 seconds. If it took ten cycles the period was 150 seconds.

The bands in the 25-degree run shown above were considerably wider and eyeball inspection suggests it would have taken at least 20 slightly offset patterns to produce those wide bands. Thus if the first pattern took say 15 seconds this run might have a period of 300 seconds. Whatever it did in that 300 seconds would repeat over and over again making it perfectly periodic.

So how might we test this possibility? I thinks its simple. First pick a point "A" where the band is widest. When the trace reaches there begin recording the actual numerical values of the variable a1 for every computational step, to say three decimal places. The first value could be labeled "a1-0". Then every time a new value of a1 is computed record it. Do this for 100 computational steps and put the results in a table. Continue running the model, and then when the trace approaches A again begin to compare the new values of a1 with a1-0. Do this at every computational step looking for an exact match. If an exact match is found see if the following 99 also match the remaining 99 in that table. If they do then the system is close to being perfectly periodic since the trace has begun follow along exactly the same path as it did once before. The length of that period would be easy to measure. Finally continue to run the model to verify that the 100 values in the table repeat periodically.

Again the point of doing this is simply to determine if what looks like quasi-periodic oscillation is actually perfectly periodic operation with longer periods than one might think.

9.9 Ideas about root cause of chaos

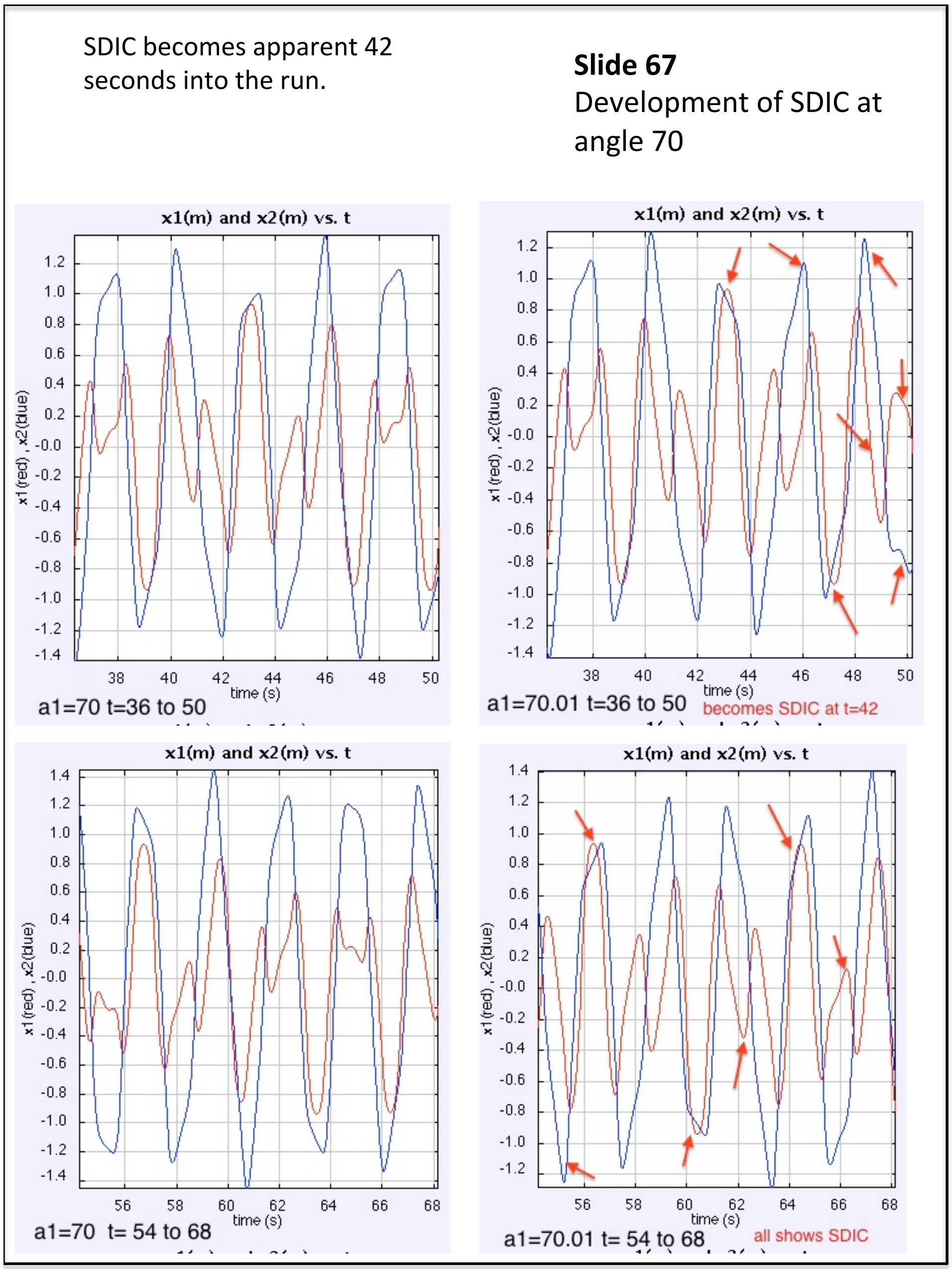
This and the next two sections are no longer a simple descriptions of behavior as observed from the simulation runs. Rather they contains several speculative thoughts.

Unfortunately I haven't discovered the root cause yet nor has anyone else to my knowledge. Gleick says even the experts don't know. I believe the explanation –if and when its found- will be relatively simple, although subtle.

Nevertheless the following two ideas may help lead someone else to an answer. If so it would apparently be the first time the root cause of chaos in some real-world system has been explained. (I feel the spider-web looking diagram used to explain the Logistics equation is inadequate or irrelevant since its not a real system.) I haven't the time or simulation models to investigate these two ideas any further. I hope someone else will.

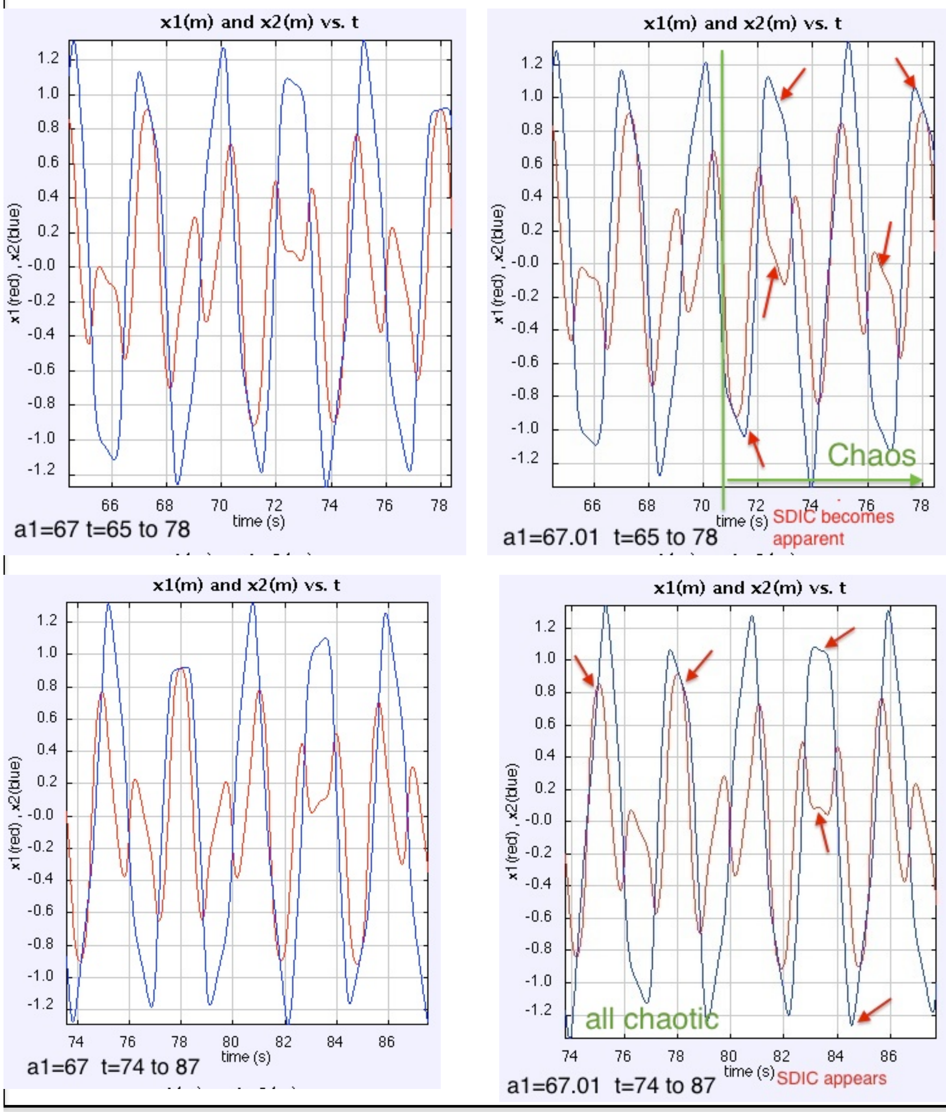
Idea #1:

We begin by noting that the double pendulum does not show visible SDIC effects immediately after the bobs are released. Instead the waveforms appear virtually identical for a while. Perhaps they do differ from the beginning but the difference grows exponentially and only becomes apparent after some time has elapsed. This clearly shows in the screenshots below. The red arrows mark where I first noticed a difference. The earlier waveform to the left looks identical at first glance but with care one can see minor differences in the heights of the wave peaks. This suggests that at some time the difference grew exponentially, so fast that it almost happened instantly. In the angle 67 run I said it happened at t=71.



SDIC first becomes apparent after about 71 seconds

Slide 69
Development of SDIC at angle 67
From t=65 to t=87 (2 of 3)



Why would a behavior change suddenly after some time had passed? It seems a tipping point of some sort might be involved. My analogy for a tipping point is rolling a ball up a dome shaped hill. If it lacks sufficient energy it reaches some altitude short of the ridge top and rolls back. With more energy it gets near the top and accelerates back slowly since the hill-top is gently rounded and almost flat at that point. Nevertheless it retraces its route back down. However if it has sufficient energy it will just reach the top and barely go over. It then has a very different future as it rolls down the far side.

Somehow in the movement of the double pendulum I suspect there is such a tipping point, or maybe many of them. If the system lacks sufficient energy it doesn't reach any of them and remains quasi-periodic and absent of SDIC. At a certain energy level it is just able to surmount the ridge, with any less it can't.

I suspect the way to better understand this is to plot the sheer force on the pivot between the two arms to see how it varies as the arms incrementally approach the point on the waveform where SDIC first becomes apparent. It will I think change exponentially and change from one arm pulling the other to one arm pushing the other.

As to why it takes time to reach the tipping point and display SDIC it may be that the system must evolve through a dance of some sort until its reached.

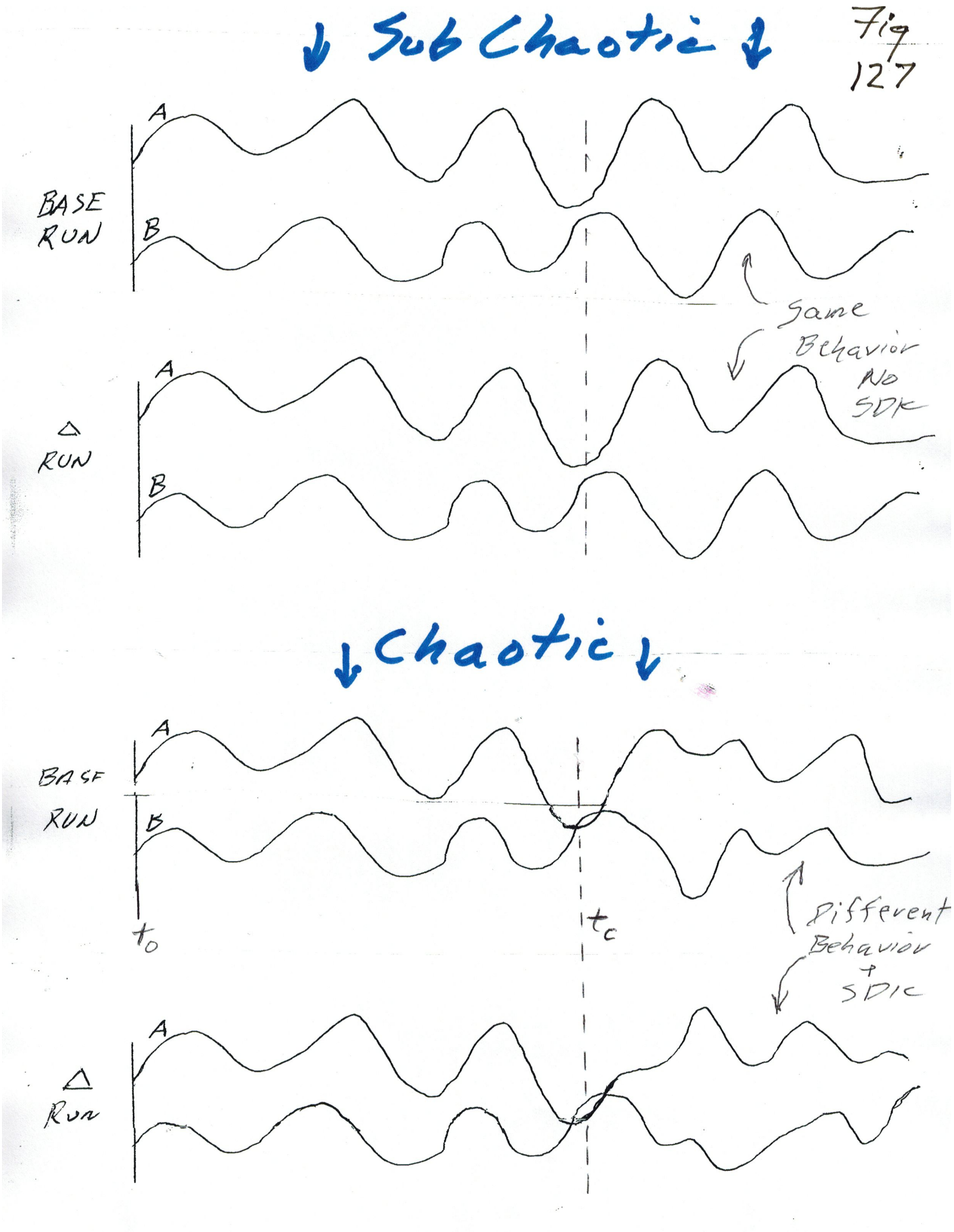
I'll let it go for now except to say this is a very simple system. We can track its movements and all the forces involved very accurately. Something simple and tangible must be causing SDIC effects to start occurring. I don't believe it's that complicated. Exponential changes near a tipping point seem core to the explanation.

I suppose an alternate explanation might be that the double pendulum is always SDIC regardless of energy level. A very small difference in initial conditions like a1 would grow and eventually cause the waveforms to differ enough to be visible. Its simply an exponential process. Very exponential. At very low energies it takes a very very long time for SDIC effects to become visible. For example some very small differences in the waveform occurred over 400 seconds into a sub-chaotic run made with a1=40 and m2=1. See details in Section 8.3. Times like this are so long that observers may not have waited for these small effects to occur. They many have prematurely judged that SDIC didn't exist, and that the system was not chaotic.

I think also of the magnetic pendulum or stars transiting a field of others. As it passes through the field of attracting bodies there are situations where it can go straight, bend left or bend right. It may have a wholly different future trajectory if it tends left or tends left. Going perfectly straight is akin to rolling a ball along a ridge-top or balancing an egg on end. Its very hard to sustain.

Idea #2 for root cause:

Figure 127 suggests a possible way to identify the root cause of chaos in the double pendulum. The key thought is that at levels of energy adequate to make the system chaotic the system reaches occasional tipping points where a very small difference in conditions at that point will cause downstream waveforms to differ significantly. At sub-chaotic levels of energy those tipping points are never reached. This takes some explanation.



Explanation of the diagram: The upper four waveforms represent sub-chaotic operation where the system has no SDIC. The lower four show chaotic behavior.

The A and B waveforms are for two variables in the system. I don't know which ones they might be. My favorite candidate for one would be the sheer force on the middle pivot because I suspect it changes from push to pull and that might represent a tipping point in the system dynamics. A second choice would be the angle between the two arms which when it passes thru zero might cause a tipping point. Possibly we would need to plot three variables.

The sub-chaotic run would be made with just a bit less energy than needed to make the system chaotic. In my runs it was perfectly periodic just below where it became chaotic but I would try to find an energy level in the narrow space between perfectly periodic and chaotic, if one exists. If not I would run it slightly below where its perfectly periodic. The chaotic runs would be made just over the E level needed to make the system SDIC and thus chaotic. I've determined those a1 or energy levels in prior sections and noted them on the slides.

The waveforms are plotted from the beginning of a simulation run marked T sub zero out beyond the time needed for the chaotic system to develop very different waveforms between the base run and the "delta" run. The base run is made with some set of initial conditions as I did with all my SDIC tests. The "delta" run is made with one of the initial conditions being slightly different. Thus if I made a1=75 in a base run I set it to 75.01 in the delta run.

The area to focus on will be at T sub c. That's where SDIC effects –if they exist- will first become obvious when comparing the base waveform with the delta waveform. This is what I did with all the SDIC tests reported earlier. I found that SDIC effects didn't appear immediately after the run was started but rather ramped up very quickly after a certain time had passed. The times ranged from 7.5 seconds to over 50 seconds. The more energy the shorter the time.

Suspected result: My premise is that something special is happening when the SDIC effects intensify in the vicinity of T sub c. And that something might appear if one compares the sub-chaotic runs with the chaotic runs. My diagram suggests a possibility. In the sub-chaotic runs the A and B waves remain apart, whereas in the chaotic runs they touch or overlap. I suspect that when they do first touch or overlap the system departs on a different behavior in the base case than in the delta case because the point at which they touch is not exactly identical. It wouldn't be because the initial conditions are not exactly identical. Its at this point that the small delta in initial conditions is magnified, or its effects are magnified. We need to examine this area under a microscope so to speak and see exactly what is happening. What it happening at T sub c that causes the downstream waves to differ so much. I envision it as when the difference between two variables approaches zero. Perhaps its analogous to a situation where a ball is rolled up to a rounded ridge top, where the slightest difference can determine whether it rolls back or over the other side. If it rolls one way we get one downstream waveform. If it rolls off the other side we get a substantially different waveform.

To me this explanation helps explain why the sub-chaotic runs experience no SDIC. In other works the base waveform is virtually identical to the delta waveform far into the future. This is what I found true in my SDIC tests. The reason is that the A and B waves never touched so no tipping point was reached. Its as though the ball was rolled up toward the ridge, but never had the energy to reach it.

It may also help explain why the chaotic waves appeared identical for a while and then suddenly diverged after 7.5 to 50 seconds in my runs. Perhaps it took the arm positions and speeds that long to reach a tipping point situation.

9.10 The essence of behavior

Each arm in a double pendulum would swing with an even rhythm like a simple pendulum because once raised and released gravity converts potential energy to kinetic energy. However since they are attached to each other the movement of arm A disturbs the movement of arm B, and vice versa. The evolution of the forces one applies to the other are very hard to visualize but they cause energy to be transferred back and forth between the two arms in a manner that continually alters what would have been their natural rhythm, position and speed. They go higher than they might otherwise, or faster, or slower, or lower. This happens at all energy levels thus making the waveforms of any of the relevant variables, like PE1, PE2, KE1, KE2, vary in a complex manner. On occasion the position and speed of arm A is such that the push or pull applied by arm B will cause it to go-over-the-top as opposed to falling back thus creating what I call a dramatic event. It's a matter of phase relationships between oscillatory motions.

9.11 Possible 4-D strange attractor

There are reasons to think the double pendulum behaves like the Lorenz waterwheel or Lorenz equations in the sense of having a strange attractor. This leads to the following hypothesis: **The double pendulum's behavior is described by a four-dimensional strange attractor similar to that that applicable to the Lorenz equations.**

It is further hypothesized that the shape of this attractor morphs as the level of energy in the system changes such that when its behavior becomes perfectly periodic the attractor converges to become a single line in 4-D space. If viewed in 3D its also a single line, as it is in the 2D ke2/a1 plots. A comparison of the ke2/a1 plots for the different perfectly periodic runs suggests that the attractor will have a different shape for each perfectly periodic situation and morph from one to the next. A long series of ke2/a1 plots made into a movie would be helpful in showing this if in fact its true.

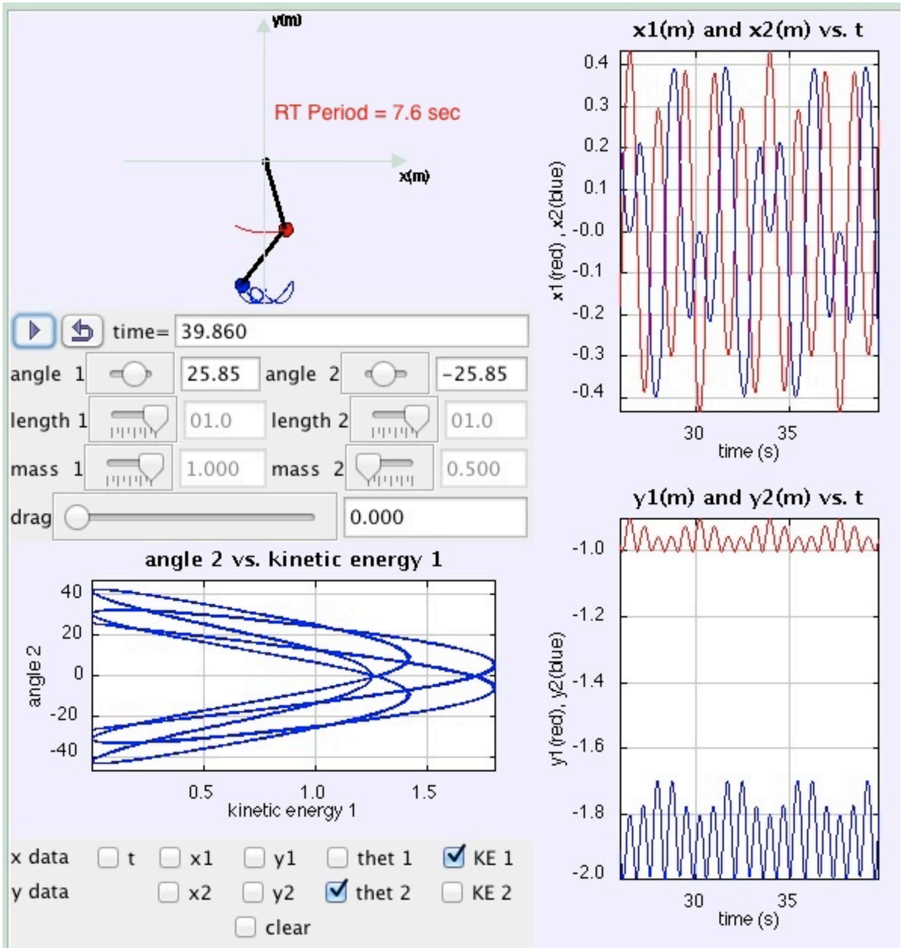
In simple English this means that the four variables in the double pendulum evolve within a 4-D band of allowed values where they repeatedly come close to prior values but never revisit them exactly. Watching a video of the Lorenz strange attractor in action illustrates what I mean by this/

The reasons for offering this hypothesis are as follows:

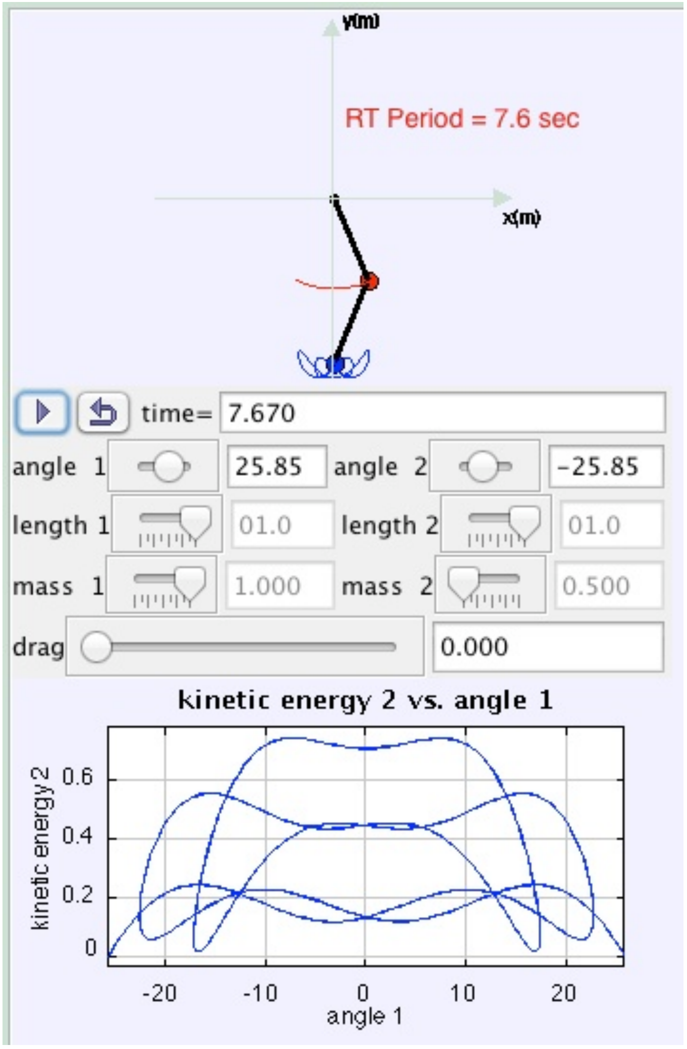
1) The notion that this is a 4-D attractor is simply because there are four interacting variables in the double-pendulum. Its phase space plot is presumably four-dimensional.
2) In the simulations presented above the trace almost never seemed to repeat exactly. The traces drifted and drew patterns offset from prior ones, but they didn't drift steadily one way or the other. Sometimes they went outside prior traces but often they fell between them. This happens in the Lorenz butterfly, which is a 3-D strange attractor. The fractal notion applies because the traces can come increasingly close to prior ones indefinitely, but they never touch. This is a fractal characteristic.
3) The behavior of the double pendulum was usually what I originally called pattern periodic in that the trace made approximately the same pattern time after time. Now I use the conventional term quasi-periodic. Similarly the trace in the Lorenz butterfly makes the same pattern (looping around and around in one wing) time after time. This needs qualification. The pattern in the double pendulum was pattern periodic at relatively low energy levels and I haven't seen the pattern (i.e.: phase space portrait) produced by the Lorenz equations at low energy. I simply suspect it shows similar spiral patterns but the orbits are confined within one wing since there are no dramatic events (i.e.: the waterwheel changing from swinging to revolving).
4) After a long time running the KE2/a1 patterns usually drifted until the trace looked like it would cover the entire available area with traces so close together they would shade the entire region and touch all possible values. This appears to be how the trace in the Lorenz butterfly behaves.

This suggests another interesting topic to explore, namely how the shape of the 4D attractor might change as the energy level in the system increases.

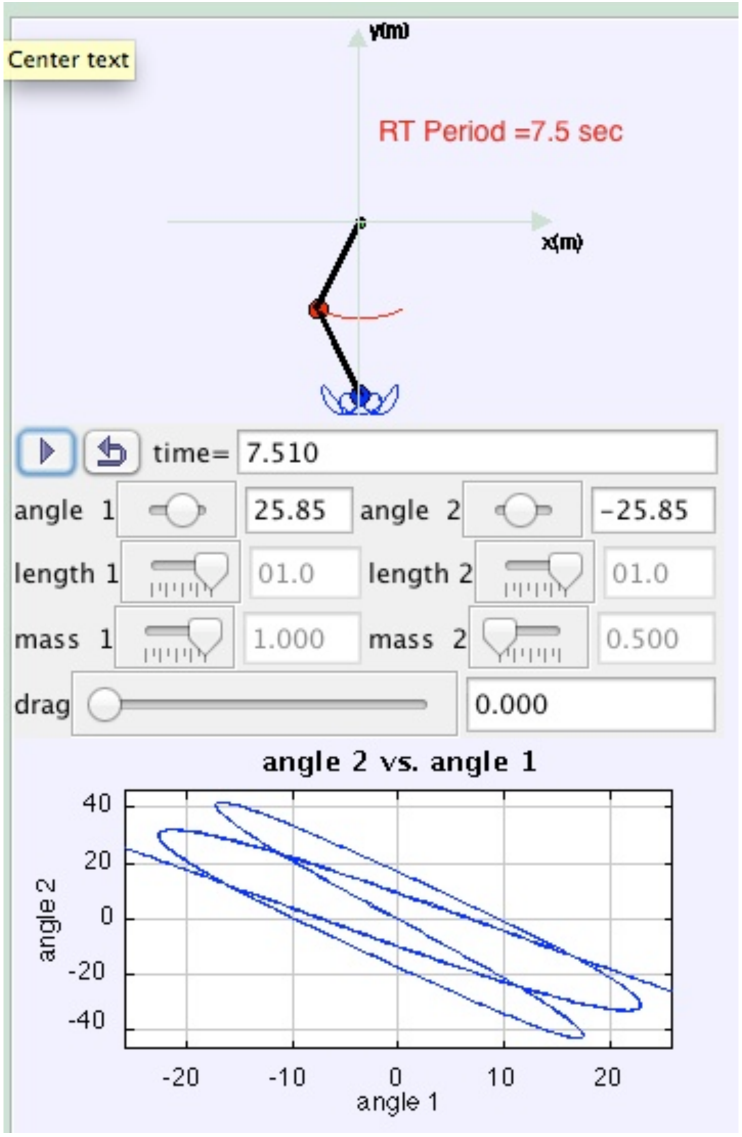
In hope to give further insight into what 4-D attractor might look like the following 2D screenshots were taken of a perfectly periodic run because it would produce a clearer picture. Each screenshot looked at the attractor from a different side if you will. The time for the trace to make one round trip around the pattern is shown in red. Even with these screenshots its very difficult to visualize what a 3-D version of this attractor would look like, much less a 4-D version. A 3-D plot would be easily to produce but this model does not provide that option.

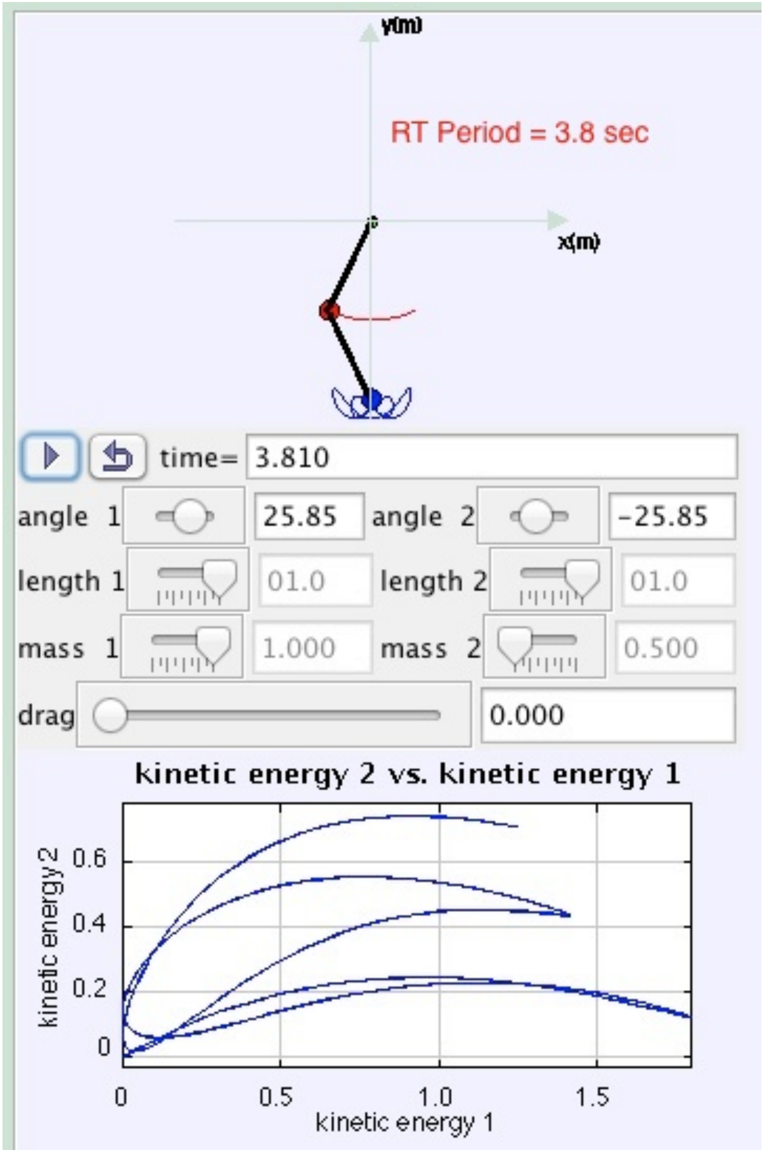


y(m)
RT Period = 7.6 sec
x(m)
time= 39.860
angle 1
25.85
angle 2
-25.85
length 1
01.0
length 2
01.0
mass 1
1.000
mass 2
0.500
drag
0.000
angle 2 vs. kinetic energy 1
angle 2
kinetic energy 1
x data
y data
t
x1
y1
thet 1
KE 1
x2
y2
thet 2
KE 2
clear
x1(m) and x2(m) vs. t
x1 (red) , x2(blue)
time (s)
y1(m) and y2(m) vs. t
y1(red), y2(blue)
time (s)

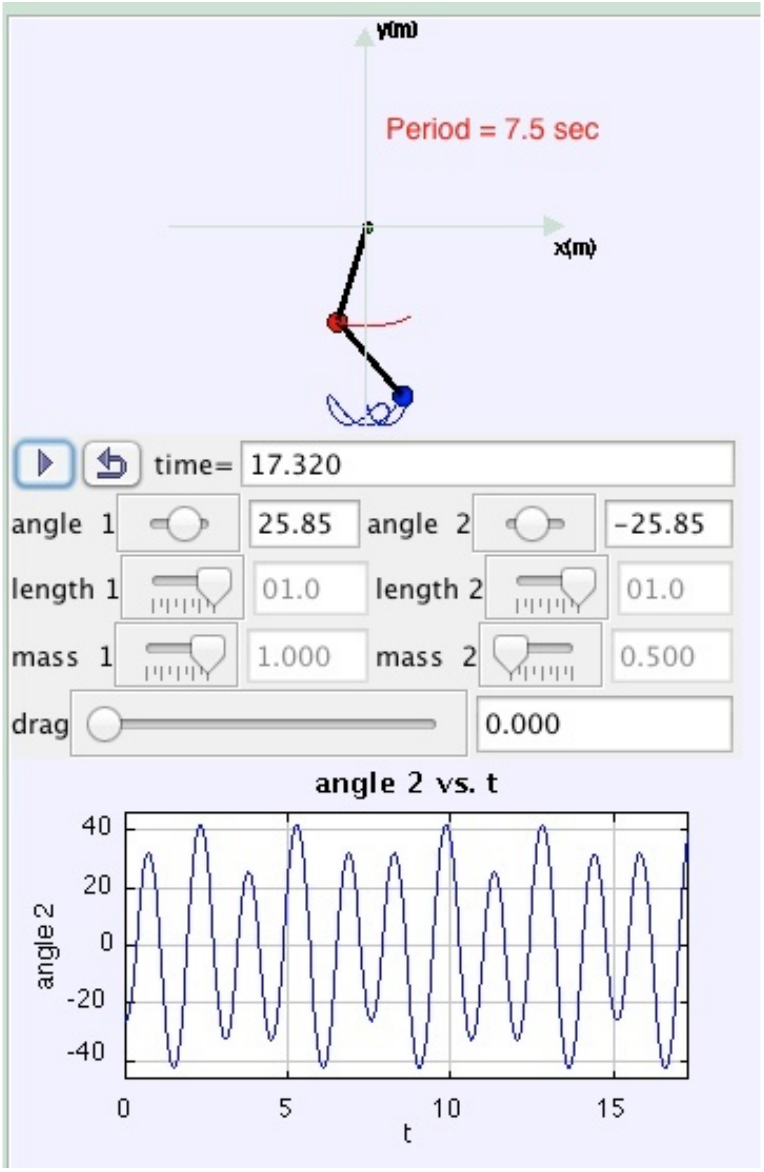


y(m)
RT Period = 7.6 sec
x(m)
time= 7.670
angle 1
25.85
angle 2
-25.85
length 1
01.0
length 2
01.0
mass 1
1.000
mass 2
0.500
drag
0.000
kinetic energy 2 vs. angle 1
kinetic energy 2
0.6
0.4
0.2
0
-20
-10
0
10
20
angle 1


Center text
y(m)
RT Period =7.5 sec
x(m)
time= 7.510
angle 1
25.85
angle 2
-25.85
length 1
01.0
length 2
01.0
mass 1
1.000
mass 2
0.500
drag
0.000
angle 2 vs. angle 1
40
20
0
-20
-40
angle 2
-20
-10
0
10
20
angle 1



The screenshot below plots angle 1 over a brief period. The peak just after t=5 reappears 7.5 seconds later at t=12.5 giving further indication that the period of this perfectly periodic run was 7.5 seconds.



Admittedly it is very hard using these 2D views to even imagine what a 3D version of that 4D strange attractor might look like.

****end of chapter 9****

Hopefully this notion of a possible 4D attractor for the double pendulum is compelling enough so others will investigate further.

9.10 goes here